A Classification of Real and Complex Nilpotent Orbits of Reductive Groups in Terms of Complex Even Nilpotent Orbits

by

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Abstract

Let $\mathfrak{g}$ be a complex, reductive Lie algebra. We prove a theorem parametrizing the set of nilpotent orbits in $\mathfrak{g}$ in terms of even nilpotent orbits of subalgebras of $\mathfrak{g}$ and show how to determine these subalgebras and how to explicitly compute this correspondence. We prove a theorem parametrizing nilpotent orbits for strong involutions of G in terms of even nilpotent orbits of complex subalgebras of $\mathfrak{g}$ and show how to explicitly compute this correspondence.

Thesis Supervisor: David Vogan
Title: Professor

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I am grateful to my parents, Birgit and Allen, and my sister Larissa for their constant support. Finally, I would like to thank my wife Anna for her endless love and encouragement.

Contents

Chapter 0

Notation

Throughout this text, we will use the following notations.

Let G be a complex Lie group and $\mathfrak{g}$ be its Lie algebra. If x is an element of G, then G^x is the centralizer of x and $\mathfrak{g}^x$ is the fixed points of $Ad(x)$.

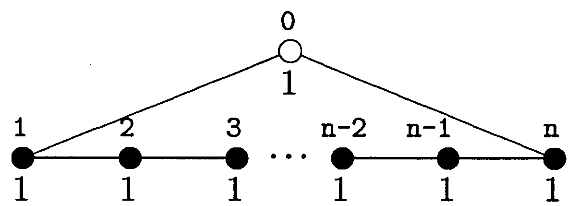
If σ is an automorphism of G, then G^σ is the fixed points of σ in G and $\mathfrak{g}^\sigma$ is the fixed points of σ in $\mathfrak{g}$.

G' is the derived subgroup of G and $\mathfrak{g}'$ is the derived subalgebra of $\mathfrak{g}$.

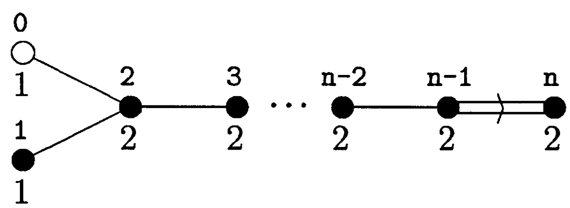
0.1 Labeled Extended Dynkin diagrams

The following are the extended Dynkin diagrams of simple complex Lie groups G. The lowest root is labeled in `typewriter` font above or to the left with the number `0`. The remaining roots in the extended Dynkin diagrams are labelled above or to the left in the `typewriter` font with the number of the vertex in the Bourbaki labeling. For a fixed root system, the root α_i corresponds to the the vertex i. The labels below the roots are the coefficients c_i so that $\alpha_0 = -\sum_{i=1}^{n} c_i\alpha_i$ and $c_0 = 1$. Thus they satisfy the formula $\sum_{i=0}^{n} c_i\alpha_i = 0$.

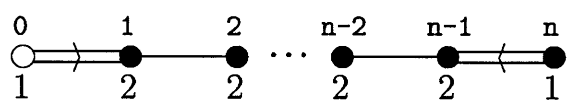
A_n



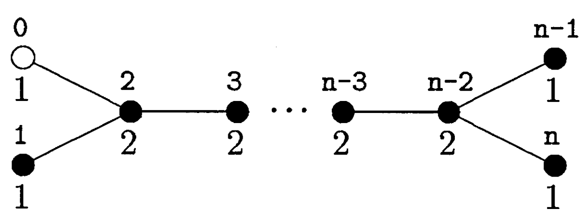
B_n

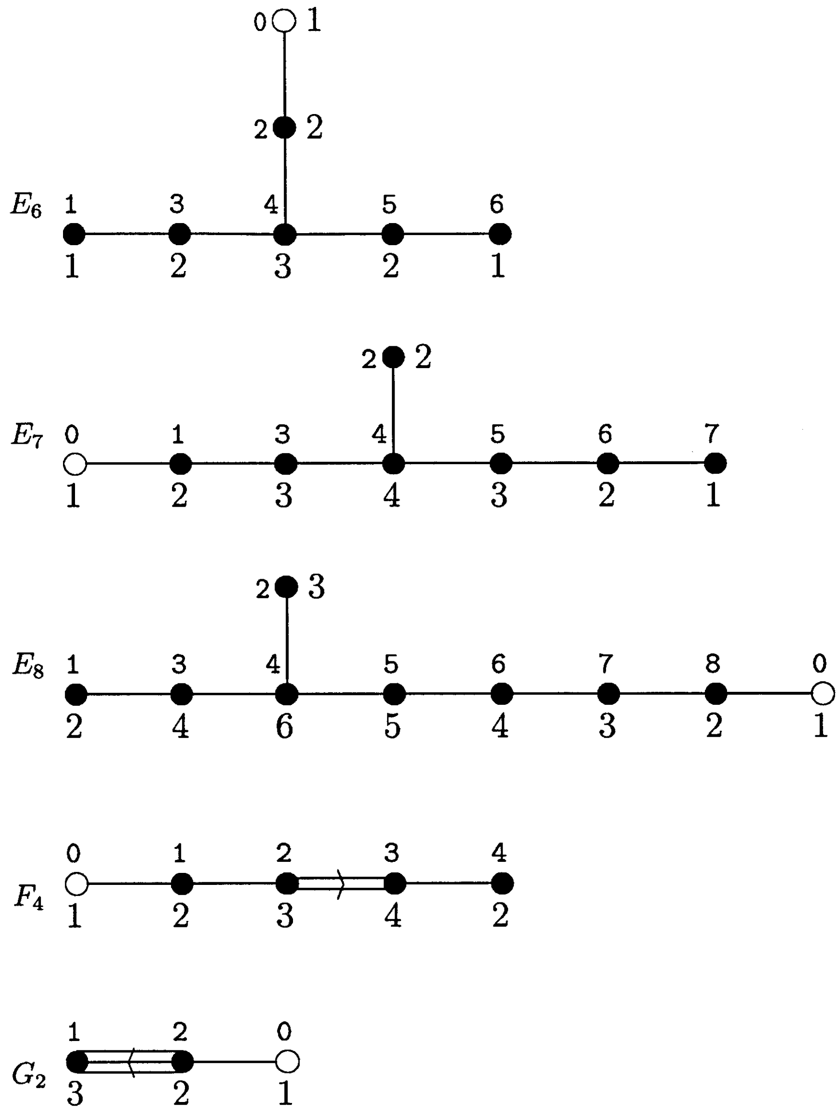


C_n



D_n




E_6
E_7
E_8
F_4
G_2

Chapter 1

Introduction

Let G be a connected, reductive group with Lie algebra $\mathfrak{g}$. The first half of this paper will concern the following theorem relating nilpotent elements in $\mathfrak{g}$ to even nilpotent elements in complex subgroups.

Theorem 1.0.1. *Suppose G is a complex connected reductive group, with Lie algebra $\mathfrak{g}$. Consider the collection $\mathcal{X}$ of all pairs (s, X) such that*

1. *$s \in G$ is an element of order 1 or 2;*

2. *$X \in \mathfrak{g}^s$ is an even nilpotent element; and*

3. *For every homomorphism ϕ: $SL(2) \to G^s$ with $d\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X$, we have $\phi(-I) = s$.*

Then the orbits of G on $\mathcal{X}$ are in one-to-one correspondence (by projection on the second factor) with nilpotent orbits of G on $\mathfrak{g}$.

The point of this theorem is the partition of the set of nilpotent elements according to the (finitely many) conjugacy classes in G of order 1 or 2. For each s, the determination of the corresponding set of nilpotent elements takes place in the (often smaller) Lie algebra $(\mathfrak{g}^s)'$. When $\mathfrak{g}^s$ is equal to $\mathfrak{g}$, then we are considering even nilpotent elements, which are somewhat easier to work with.

Section 2 is devoted to making the classification in this theorem explicit. First we discuss the classification of elements of order 2 using the method of Borel-de Siebenthal [3]. We compute the centralizer of each order two element, then (for each simple $\mathfrak{g}$ separately), explain how to finish the classification of nilpotent element.

Section 3 is devoted to the classification of nilpotent elements in real Lie algebras. We first use the idea of Cartan to replace questions about real forms by questions about holomorphic involutions (the Cartan involutions). We therefore begin with the extended group G^Γ attached to an inner class of holomorphic involutions containing γ.

Theorem 1.0.2. *Suppose G^Γ is an extended group for the complex reductive group G, corresponding to an inner class of holomorphic involutions. Consider the collection $\mathcal{Y}$ of all pairs (y, X) such that*

1. *$y \in G^\Gamma \backslash G$;*
2. *$X \in \mathfrak{g}^y$ is an even nilpotent element; and*
3. *for every homomorphism $\phi\colon SL(2) \to (G^y)'$ with $d\phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X$, we have $y^2\phi(-I) \in \mathcal{Z}(G)$.*

Then the orbits of G on $\mathcal{Y}$ are in one-to-one correspondence (by sending y to $y\phi \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$) with equivalence classes of nilpotent orbits for strong involutions of (G, γ) corresponding to the same inner class of holomorphic involutions.

The notion of equivalence classes of nilpotent orbits for strong involutions of (G, γ)will be explained in detail in section 3.1. Again the point of the theorem is the partition of the set of equivalence classes of nilpotent orbits for strong involutions of (G, γ) according to the (finitely many) G-orbits in $G^\Gamma \backslash G$ of elements y for which y^4 is in the center of G. The G orbits of such elements are closely related to automorphisms of G of order dividing 4. We discuss the Kac classification of finite order automorphisms as it pertains to classifying the G-orbits of such elements. We discuss, in several examples, how to compute this bijection explicitly, and how to find all equivalence classes of nilpotent orbits for strong involutions from the even complex nilpotent orbits.

Chapter 2

Complex Groups

2.1 Proof of Nilpotent Orbit Correspondence for Complex Groups

We will first review what we will later need about nilpotent orbits in reductive Lie algebras.

Let G be a connected, reductive complex Lie group with Lie algebra $\mathfrak{g}$. Let T be a Cartan subgroup with Lie algebra $\mathfrak{t}$. Let $(R, X, R^\vee, X^\vee)$ be the root datum of (G,T) with simple roots Π. An element $X \in \mathfrak{g}$ is nilpotent if $\mathrm{ad}(X)$ is a nilpotent automorphism of $\mathfrak{g}$ and $X \in \mathfrak{g}'$. A nilpotent orbit is a conjugacy class of nilpotent elements in $\mathfrak{g}$. There is a correspondence between nilpotent orbits in $\mathfrak{g}$ and unipotent conjugacy classes in G.

Let

$$H_{SL(2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_{SL(2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y_{SL(2)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

A standard triple (H, X, Y) is any triple of the form $(\phi(H_{SL(2)}), \phi(X_{SL(2)}), \phi(Y_{SL(2)}))$, where $\phi\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ is a homomorphism. One way to understand a nilpotent element X is by understanding the possible standard triples it is contained in.

Theorem 2.1.1. *(Jacobson-Morozov, Kostant, Mal'cev) Any nilpotent element $X \in \mathfrak{g}$ is contained in a standard triple (H, X, Y). In fact, that nilpotent triple is unique up to conjugation by G^X. Any two standard triples with the same semisimple element H are also* *conjugate.*

This theorem implies that the conjugacy class of the nilpotent element X is determined by the conjugacy class of any of the (mutually conjugate) elements H so that (H, X, Y) is a standard triple.

In order to describe the nilpotent orbit of X, we describe the conjugacy class of the possible elements H. Every semisimple element is conjugate to a unique element of the maximal torus satisfying $\alpha(H) \geq 0$ for every simple root α, and these numbers $\alpha(H)$ determine the conjugacy class. This element H is called the distinguished semisimple element corresponding to the nilpotent orbit. This information is encoded particularly nicely in the weighted Dynkin diagram which attaches to each simple root in the Dynkin diagram α the value $\alpha(H)$, called a weight. Using this theorem, one can assign any element nilpotent orbit a unique weighted Dynkin diagram.

The weighted Dynkin diagrams associated to nilpotent orbits satisfy a strong constraint.

Theorem 2.1.2. *The weights in the weighted Dynkin diagram associated to a nilpotent orbit are all* 0, 1 *or* 2.

Not all weighted Dynkin diagrams with only 0, 1, and 2 actually correspond to nilpotent orbits. For each complex semisimple group, the set of weighted Dynkin diagrams corresponding to nilpotent orbits has been determined.

We will discuss even nilpotent orbits, a smaller set of orbits, which we will show determines the remainder of the nilpotent orbits. Even nilpotent orbits are those orbits with a weighted Dynkin diagram consisting only of 2's and 0's.

Because $SL(2, \mathbb{C})$ is simply connected, there is a bijection between complex-linear Lie algebra homomorphisms $\mathfrak{sl}_2 \to \mathfrak{g}$ and complex group homomorphisms $SL(2, \mathbb{C}) \to G$, where the former is the differential of the latter. As a consequence, we find the following proposition.

Proposition 2.1.3. *There is a bijection between conjugacy classes of homomorphisms* $\phi\colon SL(2, \mathbb{C}) \to G$ *and nilpotent orbits, where the nilpotent orbit corresponding to a conjugacy class of homomorphisms* $G \cdot \phi$ *is* $G \cdot d\phi(X_{SL(2)})$.

From this proposition, it follows that the conjugacy class of $d\phi(X_{SL(2)})$ (the nilpotent orbit associated to ϕ) determines the conjugacy class of $\phi(-I)$. This conjugacy class of elements of order 1 or 2 is an invariant of the nilpotent orbit that we will discuss extensively.

Even nilpotent orbits are characterized in terms of this invariant:

Proposition 2.1.4. *Let G be a semisimple, simply connected, Lie group with maximal torus T and root system R. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the corresponding Lie algebras. Let $\phi : SL(2,\mathbb{C}) \to G$ be the homomorphism corresponding to a nilpotent orbit $\mathcal{O}$. Then $\phi(-I)$ must be an element of order 2 and $\phi(-I)$ is central if and only if the orbit is even.*

Proof. For the first part, note that $\phi(-I)^2 = \phi(I) = I$.

For the second part, we will use the condition that an element $\exp(A) \in G$ is central if and only if $\alpha_i(A) \in 2\pi i\mathbb{Z}$ for every simple root α_i. Let (H, X, Y) be a standard triple in $\mathfrak{g}$ with H in the maximal torus $\mathfrak{t}$ and H dominant with respect to the simple roots Π, so that the values $\alpha_i(H)$ are the values in the weighted Dynkin diagram for $\mathcal{O}$.

In $SL(2,\mathbb{C})$, the exponential map satisfies $\exp(i\pi H) = -I$, so that

$$\phi(-I) = \phi(\exp(\pi i H)) = \exp(\pi i d\phi(H)).$$

Then $\phi(-I)$ is central if and only if $\alpha_i(d\phi(H))$ is an even integer for every simple root α_i. Hence the orbit $\mathcal{O}$ is even if and only if $\phi(-I)$ is in the center of G. □

The following lemmas illustrate how much information about a homomorphism ϕ is contained in the order 1 or 2 element $\phi(-I)$.

Lemma 2.1.5. *Let ϕ: $SL(2,\mathbb{C}) \to G$ be a homomorphism and let $s = \phi(-I)$. Then the image of ϕ is in G^s.*

Proof. The element $-I$ is in the center of $SL(2,\mathbb{C})$, so for any $g \in SL(2,\mathbb{C})$,

$$\phi(g)s = \phi(g(-I)) = \phi(-Ig) = s\phi(g).$$

□

Let $(G^s)'$ be the derived subgroup of G^s.

Lemma 2.1.6. *Let E be a Lie group and let $\phi\colon SL(2,\mathbb{C}) \to E$ be a homomorphism. Then the image of ϕ is contained in the derived group E'.*

Proof. The group $SL(2,\mathbb{C})$ is semisimple and hence is its own derived subgroup. Any element $g \in SL(2,\mathbb{C})$ is a product of elements of the form $h_1 h_2 h_1^{-1} h_2^{-1}$. Hence, $\phi(g)$ is a product of elements $\phi(h_1)\phi(h_2)\phi(h_1^{-1})\phi(h_2^{-1})$ as well, so that $\phi(G)$ is in the derived subgroup E'. □

The following theorem characterizes nilpotent orbits in G in terms of even nilpotent orbits of subgroups.

Theorem 2.1.7. *The mapping $(s, X) \mapsto X$ induces a bijection from the pairs (s, X) modulo conjugation by G, where*

1. *$s \in G$ is an element of order 1 or 2;*
2. *X is an even nilpotent element in $\mathfrak{g}^s$; and*
3. *For every homomorphism $\phi\colon SL(2,\mathbb{C}) \to (G^s)'$ with $d\phi(X_{SL(2)}) = X$, the element $\phi(-I) = s$.*

to the nilpotent orbits of G.

Proof. The mapping is G-equivariant (under the conjugation action of G) so it induces a map on G orbits.

The mapping takes a pair (s, X) to a nilpotent element X of G^s. Any nilpotent element of G^s is also a nilpotent element of G, so the induced map takes a G-orbit of pairs (s, X) to a nilpotent orbit of G.

If X is a nilpotent element in G, then by Proposition 2.1.3, there is a homomorphism $\phi\colon SL(2,\mathbb{C}) \to G$ such that $\phi(X_{SL(2)}) = X$. Let $s = \phi(-I)$. We will show (s, X) satisfies the three properties listed. By Proposition 2.1.4, s has order 2. By Lemma 2.1.5, the image of ϕ is contained in G^s. By Lemma 2.1.6, the image of ϕ is contained in $(G^s)'$. Therefore X is a nilpotent element in G^s. Since $s \in \mathcal{Z}(G^s)$, Proposition 2.1.4 implies that X is an even nilpotent element in $\mathfrak{g}^s$. Let $\phi_1 \colon SL(2,\mathbb{C}) \to G^s$ be another homomorphism with $d\phi_1(X_{SL(2)}) = X$. Then by Theorem 2.1.1, the homomorphism ϕ_1 is conjugate to ϕ in G^s. Thus $\phi_1(-I)$ is conjugate to s in G^s. Since $s \in \mathcal{Z}(G^s)$, it follows that $\phi_1(-I) = s$. Thus, the pair (s, X) satisfies the properties listed and X is in the image of the mapping. Hence $G \cdot X$ is in the image of the induced map and the mapping is surjective. Thus the induced map is surjective.

Let (s_1, X_1) and (s_2, X_2) be two pairs as in the theorem and suppose that $g \cdot X_2 = X_1$. By Proposition 2.1.3, there are homomorphisms $\phi_1 \colon SL(2,\mathbb{C}) \to G^{s_1}$ and $\phi_2 \colon SL(2,\mathbb{C}) \to G^{s_2}$ so that $d\phi_1(X_{SL(2)}) = X_1$ and $d\phi_2(X_{SL(2)}) = X_2$. Then $\phi_1(-I) = s_1$ and $\phi_2(-I) = s_2$. Then

$$d(g \cdot \phi_2)(X_{SL(2)}) = d\phi_1(X_{SL(2)})$$

and by Theorem 2.1.3,

$$\phi_1 = h \cdot (g \cdot \phi_2)$$

for some $h \in G$. Then

$$(s_1, X_1) = hg \cdot (s_2, X_2)$$

and pairs (s_1, X_1) and (s_2, X_2) are conjugate. Hence, the induced mapping is injective. $\square$

2.2 Conjugacy Classes of order 2 elements in complex groups

For the remainder of this chapter, we will assume that G is a simple, semisimple, simply connected, complex Lie group and explicitly describe the bijection in Theorem 2.1.7 in that context. Let n be the rank of G, let T be a maximal torus of G with Lie algebra $\mathfrak{t}$ and let $W = N(T)/T$ be the corresponding Weyl group. Let $(X, R, X^\vee, R^\vee)$ be a root datum of G, and let $P = \mathbb{Z}R$ and $P^\vee = \mathbb{Z}R^\vee$ denote the root lattice and the coroot lattice, respectively. Let Q and $Q^\vee$ be dual to $P^\vee$ and P, respectively. As G is simply connected, $P^\vee = X^\vee$ and $Q = X$. Note that, with this choice of notation, $P^\vee$ is not dual to P and $Q^\vee$ is not dual to

Q. Let Π be a set of simple roots of G, numbered according to the Bourbaki labeling. We have included an appendix with these labels. Let $H_i \in \mathfrak{t}$ be the fundamental coweight for the simple root α_i, the element of $\mathfrak{t}$ satisfying $\alpha_j(H_i) = \delta_{ij}$ for $1 \leq i \leq n$. Let $\alpha_0 = -\sum_{i=1}^{n} c_i\alpha_i$ be the lowest root of (G, T) with respect to Π. Let $c_0 = 1$, so that $\sum_{i=0}^{n} c_i\alpha_i = 0$.

The mapping $e(H) = \exp(2\pi i H)$ takes the Lie algebra $\mathfrak{t}$ to the torus T and is surjective. The kernel of the map e is the coroot lattice $P^\vee$. Since every element of G is in a maximal torus conjugate to T, it follows that every element of G is conjugate to some element of T. Two elements of T are conjugate if and only if they are conjugate by the Weyl group, so the mapping e is a bijection from $\mathfrak{t}/W_{\text{aff}}$ to the set of semisimple conjugacy classes in G, where $W_{\text{aff}} = W \ltimes (P^\vee)$ is a group of isometries of $\mathfrak{t}$ generated by W and the coroot lattice acting by translation. One fundamental domain for $\mathfrak{t}/W_{\text{aff}}$ is the subset of $\mathfrak{t}$ satisfying the conditions

1. $\Re(\alpha_i(H)) \geq 0$ for each simple root α_i $(1 \leq i \leq n)$ and

2. $\Re(\alpha_0(H)) \leq 1$ for the highest root α_0.

This region is called the fundamental alcove. A more thorough discussion of this is in [5].

Proposition 2.2.1. *The non-identity central elements in G are each of the form* $\exp(\pi i H)$, *where $\alpha_i(H) = 2$ for one simple root α_i with $c_i = 1$ and $\alpha_j(H) = 0$ for every other root α_j.*

Proof. Any central element is the entirety of a conjugacy class in G. Thus it must be uniquely of the form $\exp(2\pi i H)$, where H is in the alcove described above, so that $\alpha_i(H) \geq 0$ for each simple root α_i and $\sum_{i=1}^{n} c_i\alpha_i(H) \leq 1$. The central elements are of the form $\exp(2\pi i H')$, where H' is in the coweight lattice, so each number $\alpha(H)$ must be integral and real. Since the numbers c_α are all positive integers, the only non-zero possibility for H is that $\alpha(H) = 1$ for exactly one simple root α which satisfies $c_\alpha = 1$ and $\beta(H) = 0$ for all other simple roots $\beta \in \Pi$. $\square$

The same technique will be used to identify elements whose square is central.

Proposition 2.2.2. *The conjugacy classes $\mathcal{S}$ of elements $s \in G$ with $s^2 \in \mathcal{Z}(G)$, but $s \notin \mathcal{Z}(G)$ each contain a unique element of the form* $\exp(\pi i H)$*, where $H \in \mathfrak{t}$ satisfies one of the two conditions:*

a) $\alpha_i(H) = 1$ for one simple root α with $c_i = 2$ and $\alpha_j(H) = 0$ for all other simple roots.

b) $\alpha_i(H) = 1$ for two simple roots α satisfying $c_i = 1$ and $\alpha_j(H) = 0$ for all other simple roots.

c) $\alpha_i(H) = 1$ for one simple root α satisfying $c_i = 1$ and $\alpha_j(H) = 0$ for all other simple roots.

Proof. Exactly one element of the conjugacy class must be of the form $\exp(\pi i H)$, so that $\alpha_i(H) \geq 0$ for each simple root α_i and $\sum_{i=1}^n c_i \alpha_i(H) \leq 2$. Since the square $\exp(\pi i H)^2 = \exp(2\pi i H)$ is central, $2\pi i H$ must be in $2\pi i Q^\vee$. Hence $\alpha(H)$ must be an integer for each simple root α. Given that $\alpha_0(H) = \sum_{i=1}^n c_i \alpha_i(H) \leq 2$, and $\alpha_i(H) \geq 0$ for every $i \in (1 \ldots, n)$, they must each be either 0, 1, or 2. The only possibilities are those enumerated in the statement of this theorem. □

In order to understand real even nilpotent orbits, this set of elements whose square is central will be important, whereas for understanding complex nilpotent orbits, we will also need to understand the conjugacy classes of elements which are actually of order 2.

In the third case enumerated in the theorem, the square of the element in question must be a non-identity central element. Only in the first two cases can the conjugacy class contain elements of order 2. The second case can only occur when there are at least 2 simple roots with $c_\alpha = 1$, which happens only groups of type A_n, D_n and E_6.

This theorem shows that there are at most finitely many conjugacy classes of order 2 elements. The group of type E_8 has extended Dynkin diagram

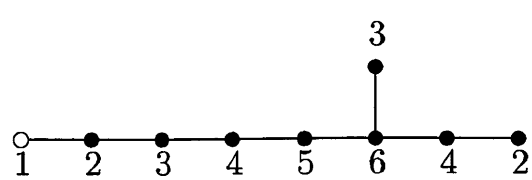


Table 2.1:

H so that $\exp(2\pi i H) \in \mathcal{Z}(G)$, $\exp(\pi i H) \notin \mathcal{Z}(G)$, and $H/2$ in fundamental alcove			
G	$H = H_i, c_i = 2$	$H = H_i + H_j, c_i = c_j = 1$	$H = H_i, c_i = 1$
A_n	none	$H_i + H_j,\ 1 \le i < j \le n$	$H_i, 1 \le i \le n$
B_n	$H_i,\ 2 \le i \le n$	none	H_1
C_n	$H_i,\ 1 \le i \le n-1$	none	H_n
D_n	$H_i,\ 2 \le i \le n-2$	$(H_1 + H_{n-1}), (H_1 + H_n), (H_{n-1} + H_n)$	H_1, H_{n-1}, H_n
G_2	H_2	none	none
F_4	H_1, H_4	none	none
E_6	H_2, H_3, H_5	$H_1 + H_6$	H_1, H_6
E_7	H_1, H_2, H_6	none	H_7
E_8	H_1, H_8	none	none

so that the simply connected complex group of type has a trivial center, and there are two conjugacy classes of order 2 elements corresponding to the right-most vertex and the second vertex from the left in the diagram.

Below, we have listed the non-central conjugacy classes whose square is central in each simply connected complex Lie group. We describe each conjugacy class in G by an element H in the alcove for G, where $\exp(\pi i H)$ is an element of that conjugacy class. All of these are clear from Proposition 2.2.2 and the coefficients c_i in the formula for the highest root.

For simply connected groups with non-trivial centers, this does not completely characterize the set of order 2 elements, as some of the conjugacy classes mentioned in the theorem may square to non-trivial central elements.

The Cartan matrix C associated to a group G has entries $C_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$.

Lemma 2.2.3. *Let H be an element of the Cartan subalgebra $\mathfrak{t}$. Let $\mathbf{a}$ be the vector with entries $a_k = \alpha_k(H)$. Then $\exp(\pi i H) = 1$ if and only if the entries of $C^{-1}\mathbf{a}$ are divisible by 2 and $\exp(\pi i H)$ has order 2 if and only if the entries of $C^{-1}\mathbf{a}$ are integral.*

Proof. The element

$$H = \sum_{j=1}^{n} b_j \alpha_j^\vee$$

for some coefficients b_j. Let $\mathbf{b}$ be the vector with entries b_j. The elements b_j satisfy

$$a_k = \sum_{j=1}^{n} b_j \langle \alpha_k, \alpha_j^\vee \rangle$$

so that

$$\mathbf{a} = \mathbf{C}\mathbf{b}$$

and

$$\mathbf{b} = \mathbf{C}^{-1}\mathbf{a}.$$

The element $\exp(\pi i H) = 1$ if and only if $H/2$ is a sum of coroots and $\exp(\pi i H)$ has order 2 if and only H is a sum of coroots. $\square$

Table 2.2:

Inverse Cartan Matrices With Non-integer Entries	
A_n	$C^{-1}_{ij} = \min(i,j) \cdot (n+1-\max(i,j))/(n+1)$
B_n	$C^{-1}_{ij} = \begin{cases} \min(i,j) & \text{if } j \neq n \\ i/2 & \text{if } j = n \end{cases}$
C_n	$C^{-1}_{ij} = \begin{cases} \min(i,j) & \text{if } i \neq n \\ j/2 & \text{if } i = n \end{cases}$
D_n	$C^{-1}_{ij} = \begin{cases} \min(i,j) & \text{if } i,j \leq n-2 \\ i/2 & \text{if } i \leq n-2,, j \geq n-1 \\ j/2 & \text{if } j \leq n-2, i \geq n-1 \\ 1/2 & \text{if } i = j \geq n-1 \\ (n-2)/2 & \text{if } i \neq j, \min(i,j) \geq n-1 \end{cases}$
E_6	$\begin{pmatrix} 4/3 & 1 & 5/3 & 2 & 4/3 & 2/3 \\ 1 & 2 & 2 & 3 & 2 & 1 \\ 5/3 & 2 & 10/3 & 4 & 8/3 & 4/3 \\ 2 & 3 & 4 & 6 & 4 & 2 \\ 4/3 & 2 & 8/3 & 4 & 10/3 & 5/3 \\ 2/3 & 1 & 4/3 & 2 & 5/3 & 4/3 \end{pmatrix}$
E_7	$\begin{pmatrix} 2 & 2 & 3 & 4 & 3 & 2 & 1 \\ 2 & 7/2 & 4 & 6 & 9/2 & 3 & 3/2 \\ 3 & 4 & 6 & 8 & 6 & 4 & 2 \\ 4 & 6 & 8 & 12 & 9 & 6 & 3 \\ 3 & 9/2 & 6 & 9 & 15/2 & 5 & 5/2 \\ 2 & 3 & 4 & 6 & 5 & 4 & 2 \\ 1 & 3/2 & 2 & 3 & 5/2 & 2 & 3 \end{pmatrix}$

The matrices C^{-1} are included in [8], section 13.2, table 1. We include them here as well, since they will be necessary for our argument.

From this and the preceding arguments, we can determine the set of non-central elements of order 2.

For example, in E_7, there are non-integral entries in row 2, so the element $\exp(\pi i H_2)$ does not square to the identity, even though its square is central. On the other hand, the elements $\exp(\pi i H_1)$ and $\exp(\pi i H_6)$ do square to the identity, since the entries in the first and sixth rows are integral.

We include a table of the order 2 elements in the simply connected, connected simple groups organized in the same categories as in Proposition 2.2.2. I will describe each conjugacy

Table 2.3:

$H \in \mathfrak{t}$ so that $\exp(\pi i H)$ is of order 2 and $H/2$ in fundamental alcove		
G	$H = H_i, c_i = 2$	$H = H_i + H_j, c_i = c_j = 1$
A_n	none	$H_i + H_{n+1-i}$, $1 \leq i \leq n/2$
B_n	H_i, $2 \leq i \leq n-1$, i even	none
C_n	H_i, $1 \leq i \leq n-1$,	none
D_n	H_i, $2 \leq i \leq n-2$, i even	$H_{n-1} + H_n$ if n odd
G_2	H_2	none
F_4	H_1, H_4	none
E_6	H_2	$H_1 + H_6$
E_7	H_1, H_6	none
E_8	H_1, H_7	none

class in G by an element H satisfying $\alpha_i(H) \geq 0$ for every simple root α_i, $\alpha_0(H) \geq -2$ for the lowest root α_0, and $\exp(\pi i H)$ in the conjugacy class.

For A_n, the element $\exp(\pi i(H_{i_1} + H_{i_2}))$ squares to the identity if and only $C^{-1}_{i_1,j} + C^{-1}_{i_2,j}$ is integral for every j. Since

$$(n+1)C^{-1}_{i,j} = \min(i,j) \cdot (n+1-\max(i,j))/(n+1) \equiv -ij \text{ (mod n+1)},$$

it follows that

$$(n+1)C^{-1}_{i_1,j} + C^{-1}_{i_2,j} \equiv -(i_1+i_2)j \text{ (mod } (n+1)),$$

so that $C^{-1}_{i_1,j} + C^{-1}_{i_2,j}$ is integral for every j if and only if $i_1 + i_2 = n+1$. Hence the elements in A_n which square to the identity are $\exp(\pi i(H_i + H_{n+1-i}))$ for $1 \leq i \leq n/2$.

For all types other than A_n, the answer is clear from inspection, checking whether a single row has only integral entries, or whether a sum of two rows has only integral entries.

In $SL(n+1, \mathbb{C})$, all order 2 matrices must be conjugate to diagonal matrices with all eigenvalues ± 1, where the conjugacy class depends on the number of -1's and 1's. There must be an even number of -1's. The Lie algebra element $H_i + H_{n+1-i}$ is the diagonal matrix whose first i entries are 1, next $n - 2i + 1$ diagonal entries are 0, and final i diagonal entries are -1. The corresponding order two element $\exp(\pi i(H_i + H_{n-i}))$ has first i diagonal entries equal -1, middle $n - 2i$ diagonal entries equal to 1 and final i diagonal entries equal to -1.

2.3 Order 2 Conjugacy Classes and the Weighted Dynkin Diagram — Odd Orbits

So far, given a simply connected, connected, simple, semisimple complex Lie group, we have determined all conjugacy classes of order 2 elements. For each homomorphism $\phi\colon SL(2,\mathbb{C}) \to G$, the element $\phi(-I)$ must lie in one of these conjugacy classes. We now discuss how to determine which one from the weighted Dynkin diagram. Recall that a homomorphism $\phi\colon SL(2,\mathbb{C}) \to G$ corresponds to an even nilpotent orbit if and only if $\phi(-I)$ is central.

First, we will discuss odd orbits and the relation between the weighted Dynkin diagram and the order 2 element corresponding to the nilpotent orbit.

Proposition 2.3.1. *Let G be a simply connected complex reductive group.*

1. *Conjugacy classes of elements of order 1 or 2 in the adjoint group $G/\mathcal{Z}(G)$ are in bijection with Weyl group orbits on $Q^\vee/2Q^\vee$. The conjugacy class of the element $\exp(\pi i H)$ is in correspondence with the orbit of the element with weights $\alpha(H)$ (mod 2).*

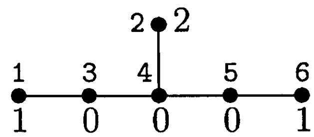
2. *Conjugacy classes of order 2 elements in G are in bijection with Weyl group orbits on $P^\vee/2P^\vee$. The element $\exp(\pi i \sum_i (a_i\alpha_i^\vee))$ is in correspondence with $\sum_i a_i\alpha_i^\vee$.*

Proof. First, note that any conjugacy class of elements in G whose square is central contains an element of the form $\exp(\pi i H)$, where H is an integral combination of coweights, so that every conjugacy class of elements of order 2 in $G/\mathcal{Z}(G)$ can be identified with a subset of $Q^\vee$. Since $\exp(2\pi i H) = 1$ in the adjoint group whenever H is an integral combination of coweights, every element of a coset in $Q^\vee/2Q^\vee$ corresponds to the same conjugacy class. Since elements of $T/\mathcal{Z}(\mathcal{G})$ are conjugate if and only if they are conjugate by the Weyl group, it also follows that Weyl group orbits on $Q^\vee/2Q^\vee$ correspond to the same conjugacy class.

Any conjugacy classes of order 2 elements contains an element of the form $\exp(\pi i H)$, where $H \in P^\vee$, so every conjugacy class of element of order 2 can be identified with a subset of $P^\vee$. Since $\exp(2\pi i H) = 1$ in the adjoint group whenever H is an integral combination of coroots, every element of the coset in $P^\vee/2P^\vee$ corresponds to the same conjugacy class.

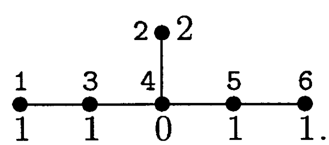
Since elements of T are conjugate if and only if they are conjugate by the Weyl group, it also follows that Weyl group orbits on $P^\vee/2P^\vee$ correspond to conjugacy classes. $\square$

When there are no order 2 elements in $\mathcal{Z}(G)$, this proposition makes it possible to identify the conjugacy classes of order 2 elements associated to a nilpotent orbit from its weighted Dynkin diagrams. For example, there is a nilpotent orbit in E_6 with the following weighted Dynkin diagram.

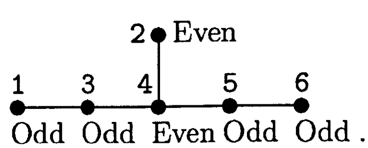


As E_6 has no order 2 elements in the center, conjugacy classes whose square is central are not interchanged by central elements, so we may apply the first half of the Proposition 2.3.1. Since $\exp(H_1/2 + H_5/2)$ was one of the order 2 elements and the vertices 1 and 5 are the only elements with 1's, the theorem immediately implies that the nilpotent order 2 element associated to this orbit is $\exp(H_1/2 + H_5/2)$.

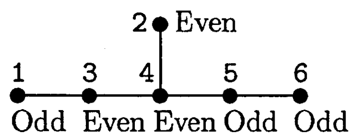
For other nilpotent orbits in E_6, it is somewhat more difficult to determine the associated order 2 conjugacy class. We will illustrate this for another nilpotent orbit of E_6, the one with Dynkin diagram



Let (X, H, Y) be the standard triple for this nilpotent orbit. We will only use the parity of the weights mod 2 of the weighted Dynkin diagram, as this determines the element of order 2, so we will simply consider the weighted Dynkin diagram as

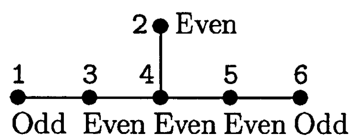


Then since conjugacy classes of order 2 elements are preserved by the Weyl group, we know that the order 2 element $\exp(\pi i H)$ is conjugate to $\exp(\pi i(s_1(H)))$, where s_1 is the reflection associated to the left vertex. $\mathbb{Z}$-weighted Dynkin diagrams correspond exactly to $\mathbb{Z}$-gradings of the root system, and $\mathbb{Z}/2\mathbb{Z}$ weighted diagrams to $\mathbb{Z}/2\mathbb{Z}$ gradings. The weighted Dynkin diagram associated to $s_1(H)$ is of the form



,

where the parity of vertex 3 was changed. Then, let s_6 be the reflection associated to the right vertex, so that the element $s_6 s_1(H)$ is of the form



.

Therefore, $\exp(\pi i(H_1 + H_6)) = \exp(\pi i(s_6 s_1(H)))$, so that $\exp(\pi i H)$ is conjugate to $\exp(\pi i(H_1 + H_6)$, one of the representatives of order 2 conjugacy classes we identified above.

Although this case does generalize somewhat, a few caveats should be pointed out.

In this case, a Weyl group element of length only 2 was needed to prove this equivalence. In other cases, much longer elements of the Weyl group are needed. Especially in those cases, it is not even clear how to generate such a Weyl group element. Of course, there are only 64 elements in $Q^\vee/2Q^\vee$, so the search is finite and not too large, but it is not clear how to proceed more efficiently. We will later discuss a method for finding the order 2 element from the Dynkin diagram in this odd case that is more systematic.

Second, we used the fact that E_6 had no order 2 elements in the center since otherwise a single conjugacy class would not correspond to a full Weyl group orbit on $Q^\vee/2Q^\vee$. In other groups, we can determine the element of the fundamental alcove associated to an order 2 element $\exp(\pi i H) \in G$ by writing H in the coroot basis and applying the second part of this proposition.

The more general way of finding the element of the fundamental alcove associated to $\exp(\pi i H)$ follows from two lemmas below. The affine Weyl group W_{aff} acts discretely and is generated by the reflections s_i and the translation $\alpha_0^\vee$.

Lemma 2.3.2. *Let $H \in \mathfrak{t}$ be a dominant element ($\alpha(H) \geq 0$ for every root α) so that $\exp(2\pi i H) = 1$ and let α_0 be the negative of the longest root. Then either H is in the fundamental alcove of $||H + \alpha_0^\vee|| < ||H||$.*

Proof. Assume H is not in the fundamental alcove. Then

$$\alpha_0(H) < -1.$$

Then

$$\begin{aligned}
||H||^2 - ||H + \alpha_0^\vee||^2 &= -(\alpha_0^\vee, \alpha_0^\vee) + 2(H, \alpha_0^\vee) \\
&= (\alpha_0^\vee, \alpha_0^\vee)\left(-1 - 2\frac{(H, \alpha_0^\vee)}{(\alpha_0^\vee, \alpha_0^\vee)}\right) \\
&= ||\alpha_0^{vee}||^2 \left(-1 - \alpha_0(H)\right).
\end{aligned}$$

Hence

$$||H||^2 - ||H + \alpha_0^\vee||^2 > 0.$$

□

Proposition 2.3.3. *Let $H^{(0)} \in \mathfrak{t}$. Then, for each i for which $H^{(i)} \in \mathfrak{t}$ is not dominant, let α_{j_i} be a simple root in Π so that $\alpha_{(j_i)}(H) < 0$ and let $H^{(j)} = s_j H^{(j-1)}$. Then all $H^{(i)}$ are distinct and contained in $\mathfrak{t}$, and for some M, $H^{(M)}$ is dominant.*

Proof. Let $d(j)$ be the distance from α_j to α_0 on the Dynkin diagram. Assign a partial ordering to the roots α_i $(0 \leq i \leq n)$ where $\alpha_i < \alpha_j$ if the distance from $d(i) < d(j)$ to α_0.

Assign a partial ordering to the Weyl group orbit $W \cdot H^{(0)}$ by letting $H' \leq H$ if for some j,

$$\begin{aligned} \alpha_i(H') &= \alpha_i(H) \text{if } d(i) \leq d(j), i \neq j \\ \alpha_j(H') &< \alpha_j(H) \end{aligned}$$

Then if $H^{(i)}$ and j_i are as in the proposition, then for $d(k) < d(j_i) - 1$, the root α_k must be perpendicular to α_{j_i} and so $\alpha_k(H^{(i)}) = \alpha_k(H^{(i+1)})$. If $d(k) = d(j_i) - 1$, then

$$\alpha_k(H^{(i+1)} = \alpha_k(H^{(i)} + \langle \alpha_j, \alpha_k^\vee \rangle \alpha_{j_i}(H^{(i)}).$$

Since $\langle \alpha_j, \alpha_k^\vee \rangle \leq 0$ whenever $j \neq k$ and is nonzero for at least one k with $d(k) < d(j_i) - 1$, we find that

$$H^{(i+1)} < H^{(i)}$$

in the partial order on $W \cdot H^{(0)}$. Thus, all of the elements $H^{(i)}$ are distinct and in the Weyl group orbit $W \cdot H^{(0)}$. As W is finite, this implies that for some M, $H^{(M)}$ is dominant. □

These two lemmas allows for the following algorithm which, given an element $H \in \mathfrak{t}$ so that $\exp(\pi i H))$ is of order 2 returns the element H' of the fundamental alcove, so that $\exp(\pi i H')$ is conjugate to $\exp(\pi i H)$:

Proposition 2.3.4. *Let $H^{(0)}$ be an element of the Cartan subalgebra $\mathfrak{t}$. Whenever $H^{(i)}$ is not dominant, let j_i be a vertex of the Dynkin diagram so that $\alpha_{j_i}(H^{(i)}) < 0$ and let $H^{(i+1)} = s_{j_i}(H^i)$. If $H^{(i)}$ is dominant but not in the fundamental alcove, let $H^{(i+1)} = H^i + \beta^\vee$. Then for some N, $H^{(N)}$ is in the fundamental alcove and $\exp(2\pi i H^{(N)})$ is conjugate to $\exp(2\pi i H^{(0)})$.*

Proof. By Lemmas 2.3.2 and 2.3.3, all the elements $H^{(i)}$ are distinct and are conjugate to $H^{(0)}$ by the affine Weyl group and we have $||H^{(i)}|| \leq ||H^{(0)}||$. The affine Weyl group acts discretely, so there are only finitely many such points. The final one, $H^{(N)}$ is dominant and in the fundamental alcove. As $H^{(N)}$ is conjugate to $H^{(0)}$ by the affine Weyl group, it follows that $\exp(2\pi i H^{(0)})$ is conjugate to $\exp(2\pi i H^{(N)})$ in G. $\square$

2.4 Order 2 Conjugacy Classes and Weighted Dynkin Diagrams - Even Orbits

We now consider even nilpotent orbits. For semisimple, simple simply connected complex groups with no order 2 elements in the center, e.g. A_n for n even, G_2, F_4, E_6, and E_8, any central order 2 element which squares to 1 must be equal to 1. For groups B_n, C_n, and E_7, the center has order 2. In that case, $\exp(\pi i H)$ must be 1 if and only if $\pi i H$ is an even integral combination of coroots, and it must be the other central element otherwise. This can be determined individually for each group.

Proposition 2.4.1. *Let $H \in \mathfrak{h}$ be an element so that $\exp(2\pi i H) = 1$ and $\exp(\pi i H)$ is central, for example, if (X, H, Y) is a standard triple associated to an even nilpotent orbit. Then $\exp(\pi i H)$ (equal to $\phi(-I)$ when (X, H, Y) is a standard triple) is the non-trivial central element of order two if the Dynkin diagram for H satisfies the following conditions:*

1. *A_n, n odd: $\alpha_{(n+1)/2}(H) = 2$.*
2. *B_n: $|\{i : \alpha_{2i-1}(H) = 2, 1 \leq i \leq (n+1)/2\}|$ odd*
3. *C_n: $\alpha_n(H) = 2$.*
4. *E_7: An odd number of $\alpha_2(H)$, $\alpha_5(H)$, and $\alpha_7(H)$ are equal to 2.*
5. *D_n, n odd: $|\{i : \alpha_{2i-1}(H) = 2, 1 \leq i \leq (n-1)/2\}|$ odd*

Proof. We use Lemma 2.2.3. Let v be the vector with entries $\alpha_i(H)$ and C^{-1} the inverse Cartan matrix. In these cases, there is only one order 2 non-trivial central element, so a Dynkin diagram for a nilpotent orbit corresponds to the non-trivial central element if it is even (so that the order 2 element is central) and $C^{-1}v$ is non-integral (so that the order

2 element is non-trivial). The conditions in the theorem are from inspecting the table of elements $C^{-1}v$. $\square$

The situation for D_{2n} is somewhat more complicated as there are more than 2 central elements of order 2. The center of Spin($4n$) is a Klein four group ([6], Chapter 10, Theorem 3.22).

Proposition 2.4.2. *Let* $G = \mathrm{Spin}(4n)$. *The non-trivial elements of* $\mathcal{Z}(G)$ *are* $\exp(2\pi i H_j)$ *for* $j = 1, 2n-1, 2n$. *Let* $H \in 2Q^\vee$.

1. *Let*

$$\begin{aligned} a &= \frac{1}{2}\sum_{j=1}^{n-1} \alpha_{2j-1}(H) \\ b &= \frac{1}{2}\alpha_{2n-1}(H) \\ c &= \frac{1}{2}\alpha_{2n}(H) \end{aligned}$$

   Then $\exp(\pi i H) = \exp(2\pi i H_1)^a \exp(2\pi i H_{2n-1})^b \exp(2\pi i H_{2n})^c$.

2. *Suppose* H *is a dominant semisimple element of a standard triple for* $\mathfrak{g}$.

   (a) *If* $\alpha_{2n}(H) = \alpha_{2n-1}(H)$ *and* $|\{j : \alpha_{2j-1}(H) = 2, 1 \le j \le n\}|$ *is even, then* $\exp(\pi i H) = I$.

   (b) *If* $\alpha_{2n}(H) = \alpha_{2n-1}(H)$ *and* $|\{j : \alpha_{2j-1}(H) = 2, 1 \le j \le n\}|$ *is odd, then* $\exp(\pi i H) = \exp(\pi i H_1)$.

   (c) *If* $\alpha_{2n}(H) \ne \alpha_{2n-1}(H)$ *and* $|\{j : \alpha_{2j-1}(H) = 2, 1 \le j \le n\}|$ *is odd, then* $\exp(\pi i H) = \exp(\pi i H_{2n-1})$.

   (d) *If* $\alpha_{2n}(H) \ne \alpha_{2n-1}(H)$ *and* $|\{j : \alpha_{2j-1}(H) = 2, 1 \le j \le n\}|$ *is even, then* $\exp(\pi i H) = \exp(\pi i H_{2n})$.

Proof. This result can also be gleaned from looking at C^{-1}. Let σ be the projection from $\mathfrak{t}$ to $\mathfrak{t}/P^\vee$. From the expression for C^{-1},

$$
\begin{aligned}
\sigma(H_{2j}) &= 0 \text{ if } j < n \\
\sigma(H_{2j-1}) &= \sigma\left(\alpha_{2n-1}^{\vee}/2 + \alpha_{2n}^{\vee}/2\right) \text{ if } j < n \\
\sigma(H_{2n-1}) &= \sigma\left(\left(\sum_{j=1}^{n-1} \alpha_{2j-1}^{\vee}/2\right) + \alpha_{2n-1}^{\vee}/2\right) \\
\sigma(H_{2n}) &= \sigma\left(\left(\sum_{j=1}^{n-1} \alpha_{2j-1}^{\vee}/2\right) + \alpha_{2n}^{\vee}/2\right).
\end{aligned}
$$

Since $\sigma(H_{2j-1}) = \sigma(H_1)$ for $1 \leq j \leq (n-2)/2$, it follows that $\exp(2\pi i H_{2j-1}) = \exp(2\pi i H_1)$. Then

$$
\exp(\pi i H) = \prod_{j=1}^{2n} \exp(2\pi i H_j)^{\alpha_j(H)/2},
$$

which implies the first part of this proposition.

For the second part, Theorem 2.1.2 and Proposition 2.1.4 imply that $\alpha_j(H)$ is equal to 0 or 2. The first part of this proposition and the group structure on the Klein four group imply the characterization of $\exp(\pi i H)$ in the second part of this proposition.

□

2.5 Centralizers of order 2 elements

For each order 2 element s in G, we will compute the centralizer G^s and the elements in the preimage of s in $\widetilde{(G^s)}'$. We will also use these results to compute the centralizer $(G/\mathcal{Z}(G))^s$ for each conjugacy class of order 2 elements $s \in G/\mathcal{Z}(G)$.

Borel-de Siebenthal theory [3] computes the centralizers of the order 2 elements as follows.

Theorem 2.5.1. *1. Let $s \in G$ be a semisimple element of the form $\exp(\pi i H)$, where $\alpha_k(H) = 1$ for one simple root α_k with $c_k = 2$ and $\alpha_i(H) = 0$ whenever $i \neq k$. The centralizer G^s is a connected semisimple group. A set of simple roots for G^s is given* *by $\{\alpha_0\} \cup \Pi \backslash \alpha_k$ and the corresponding Dynkin diagram is the extended Dynkin diagram of G with respect to Π with the vertex α_k and all edges meeting α_k removed.*

2. *Let $s \in G$ be a semisimple element of the form $\exp(\pi i H)$, where $\alpha(H) = 1$ for two simple roots α_k, α_l with $c_k = c_l = 1$ and $\beta(H) = 0$ for all other simple roots β. Then G^s is a connected reductive group. A set of simple roots for G^s is $\{\alpha_0\} \cup \Pi \backslash \{\alpha_k, \alpha_l\}$ and the corresponding Dynkin diagram is the extended Dynkin diagram of G with respect to Π with the vertices α_k and α_l removed along with all edges meeting them.*

3. *Let $s \in G$ be a semisimple element of the form $\exp(\pi i H)$, where $\alpha(H) = 1$ for one simple root α_k with $c_k = 1$ and $\alpha_i(H) = 0$ whenever $i \neq k$. The centralizer G^s is a connected semisimple group. A set of simple roots for G^s is given by $\Pi \backslash \alpha_k$ and the corresponding Dynkin diagram is the Dynkin diagram of G with respect to Π with the vertex α_k and all edges meeting α_k removed.*

In each of the cases enumerated in the theorem, we will refer to the basis roots given here in Theorem 2.5.1 as Π^s.

Let $s \in G$ be an element of order 2. Below we will discuss the relationship between the fundamental coweights for (G, Π) and (G^s, Π^s). We will divide into the two cases depending on s.

First, suppose s is an order 2 element $\exp(\pi i H_k)$, where $c_k = 2$. The coweight basis L_j of $\mathfrak{t}$ for the pair (G^s, T), is defined by the relations $\alpha_i(L_j) = \delta_{ij}$ for $i, j \in \{0, \ldots, k-1, k+1, \ldots, n\}$. The change of basis between $\{H_i\}$ and $\{L_j\}$ follows from the equation

$$\sum_{i=0}^{n} c_k \alpha_k(H) = 0.$$

They are given by

$$
\begin{aligned}
H_k &= -2L_0 \\
H_i &= L_i - c_i L_0 \text{ if } i \neq k \\
L_i &= H_i - \frac{1}{2} c_i H_k \text{ if } i \neq 0.
\end{aligned}
$$

Second, suppose s is an order 2 element $\exp(\pi i(H_k + H_l))$, where $c_k = c_l = 1$. In this case, the subgroup G^s is not semisimple; it has a one-dimensional center. We will postpone a discussion of how to compute the change of basis from the fundamental coweights of (G, T) to those of (G^s, T) to Proposition 3.2.4. For all of the computations we will do to compute the bijection in Theorem 2.1.7, the following observation about these changes of bases will be sufficient. Let τ be an order 2 automorphism of the Dynkin diagram of G. Let γ_τ be an order 2 automorphism of G preserving T with $\gamma_\tau(H_i) = H_{\tau(i)}$. In all cases, such a Dynkin diagram automorphism exists, $H_{\tau(k)} = H_l$, $\gamma_\tau(L_0) = L_0$ and $c_i = c_{\tau(i)}$. We can solve the equation

$$\sum_{i=0}^{n} c_i \left(\alpha_i + \tau(\alpha_i)\right)(H) = 0$$

to find that

$$
\begin{aligned}
H_k + H_l &= -2L_0 && (2.1) \\
H_i + H_{\tau(i)} &= L_i + L_{\tau(i)} - 2c_i L_0 \text{ if } i \neq k, l && (2.2) \\
L_i + L_{\tau(i)} &= H_i + H_{\tau(i)} - c_i(H_k + H_l) \text{ if } i \neq 0. && (2.3)
\end{aligned}
$$

Let $s = \exp(\pi i H)$, where H is as in Proposition 2.2.1. Let $\widetilde{(G^s)'}$ be the universal cover of $(G^s)'$ and let $\sigma : \widetilde{G^s} \to G^s$ be the projection. The following proposition enumerates the central elements of $(\widetilde{G^s})'$.

Proposition 2.5.2. *Let $s = \exp(\pi i H_s)$, where $H_s/2$ is in the fundamental alcove. For each component C of the Dynkin diagram of $\widetilde{(G^s)'}$ presented in Theorem 2.5.1, denote the coefficients of the highest root of the corresponding group $\alpha_C = n_{iC}\alpha_i$. Any element of*

$\mathcal{Z}(\widetilde{(G^s)'})$ is of the form $\exp_{\widetilde{G^s}}(2\pi i H)$ where $H = \sum_C \sum_{i\in C} a_i L_i$ and $\sum_{i\in C} a_i L_i$ is of the form presented in Proposition 2.2.1 for each component C of the Dynkin diagram.

Let S be the set of elements x in $\widetilde{(G^s)'}$ so that $\sigma(x) = s$. The elements of S can be determined from the following lemma.

Lemma 2.5.3. *Let $H_s \in \mathfrak{t}$ be a Lie algebra element and let $s = \exp(\pi i H_s)$. Let $H_x \in \mathfrak{t}$. The element $x = \exp_{\widetilde{G^s}}(2\pi i H_x)$ satisfies $\sigma(x) = s$ if and only if $H_x - H_s/2 \in P^\vee$.*

Proof. The element $\sigma(x) = s$ if and only if

$$\sigma(x)s^{-1} = \exp(2\pi i(H_x - H_s/2)) = I,$$

which is true if and only $H_x - H_s/2 \in P^\vee$. ☐

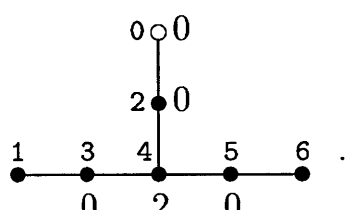
Note that $|S| = \frac{|\mathcal{Z}(\widetilde{(G^s)'})|}{|\mathcal{Z}(G^s)|}$, and in every case, this is either 1 or 2. In other words, the cover $\widetilde{G^s}$ is either a single or double cover of G^s. From the changes of bases above, it follows that s is always of the form $\exp(-2\pi i L_0)$. Since $s = s^{-1}$, this implies that s is always of the form $\exp(2\pi i L_0)$. This determines one element of S.

Below is a table of conjugacy classes of order 2 elements and the corresponding set S, where an element of S is described by an element H of $\mathfrak{t}$ written as a linear combination of fundamental coweights for G^s, so that $\exp_G(2\pi i H) = s$ and H is in the fundamental alcove for $(G^s)'$.

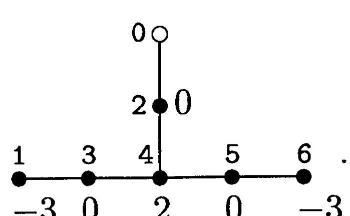
Table 2.4:

Conjugacy Classes of Order 2 Elements and Related Data			
G	$s = \exp(\pi i H)$	G^s type	$\sigma^{-1}(s)$
$A_n, n \geq 1$	$H_i + H_{n+1-i}, 1 \leq i < n/2$	$A_{2i-1} \times A_{n-2i}$	L_0
$B_n, n \geq 2$	$H_i, 4 \leq i \leq n-1, i$ even	$D_i \times B_{n-i}$	$L_0, L_0 + L_{i+1}$
$C_n, n \geq 3$	$H_i, 1 \leq i \leq n-1$	$C_i \times C_{n-i}$	L_0
$D_n, n \geq 4$ even	$H_i, 1 \leq i \leq n-4, i$ even	$D_i \times D_{n-i}$	$L_0, L_1 + L_{i+1}$
E_6	H_2	$A_5 \times A_1$	L_0, L_4
	$H_1 + H_6$	D_5	L_0
E_7	H_1	$D_6 \times A_1$	L_0, L_3
	H_6	$D_6 \times A_1$	$L_0, L_2 + L_7$
E_8	H_1	D_8	L_0, L_2
	H_8	$E_7 \times A_1$	L_0, L_6
F_4	H_1	$C_3 \times A_1$	L_0, L_2
	H_4	B_4	L_0
G_2	H_2	$A_1 \times A_1$	L_0, L_1

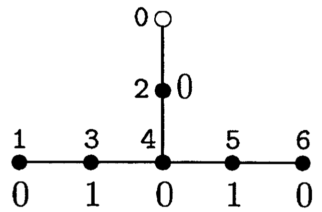
We have identified that if $\mathcal{O}$ is an odd nilpotent orbit and ϕ: $SL(2,\mathbb{C}) \to G$ is a corresponding homomorphism and (X, H, Y) the standard triple, then the orbit $\mathcal{O}$ intersects the centralizer $G^{\phi(-I)}$. The Dynkin diagram for $\mathcal{O}$ is given from that of $\mathcal{O} \cap G^{\phi(-I)}$ by a change of basis from the basis $\{L_j\}$ to the basis $\{H_i\}$. For example, consider the group $G = E_6$ with central element $s = H_1 + H_6$. Then $G^s = D_5$. One nilpotent orbit of D_5 has the Dynkin diagram



By equation 2.2, this corresponds to the element $2L_4 = 2H_4 - 3H_1 - 3H_6$, which is associated to the order 2 element $\exp(2\pi i L_0)$ from our classification of central elements in D_n. The element $2H_4 - 3H_1 - 3H_6$ corresponds to the Dynkin diagram



By applying the algorithm in Lemma 2.3.3, we find that the diagram of the dominant element of $\mathfrak{t}$ conjugate to this one is



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2.6 Correspondence in $SL(n, \mathbb{C})$

We will show how the bijection in Theorem 2.1.7 occurs in $SL(n, \mathbb{C})$.

The elements of order 2 are diagonalizable and have only eigenvalues 1 and -1. Therefore, they are conjugate to diagonal matrices with only 1 and -1 on the diagonal, where the number of -1 entries must be even due to the determinant constraint. The only central element of order 2 is $-I$, which is only in the group when n is even.

The non-central elements s of order two have centralizers $(GL(V_1) \times GL(V_{-1})) \cap SL(n)$, where V_1 is the 1-eigenspace of s and V_{-1} is the -1 eigenspace of s, which must be even-dimensional.

In $SL(n, \mathbb{C})$, nilpotent orbits are in bijection with partitions. Homomorphisms from $SL(2, \mathbb{C})$ to $SL(n, \mathbb{C})$ are n-dimensional representations of $SL(2, \mathbb{C})$. Let V_n be the n-dimensional representation of $SL(2, \mathbb{C})$. Then, then the partition $\mathbf{p}$ corresponds to any representation of the form $\oplus_{p \in \mathbf{p}} V_p$. For any irreducible representation $\phi_n \colon SL(2, \mathbb{C}) \colon SL(n, \mathbb{C})$, the matrix $d\phi_n(H_{SL(2)})$ is diagonalizable with eigenvalues $n-1, n-3, \ldots, -3-n, -1-n$. Hence $\phi_n(-I) = \exp(\pi i H_{SL(2)})$ has all eigenvalues equal to 1 (resp. -1) if n is odd (resp. even).

Let T be the diagonal maximal torus of $SL(n, \mathbb{C})$ with Lie algebra $\mathfrak{t}$. Let E_{ij} be the n by n matrix with (i, j) entry equal to 1 and all other entries equal to $-$. Define $e_i \in \mathfrak{t}^*$ to be the linear functional so that $e_i(E_{jj}) = \delta_{ij}$. Then simple roots consistent with the Bourbaki labeling are given by

$$\alpha_i = e_i - e_{i+1} \text{ for } 1 \leq i \leq n-1.$$

Either by direct evaluation or, from Proposition 2.2.1 (since any two formulas for the only non-trivial central element must agree), we find that $\exp(2\pi i H_{(n+1)/2}) = -I$ when n is odd.

For m even, the element $\exp(\pi i(H_{m/2} + H_{n-m/2})$ is diagonal with an m-dimensional -1-eigenspace V_1 and an $n-m$-dimensional 1-eigenspace V_{-1}.

Proposition 2.6.1. *Let $G = SL(n, \mathbb{C})$. Let $\phi \colon SL(2, \mathbb{C}) \to SL(n, \mathbb{C})$ be a homomorphism corresponding to a partition $\mathbf{p}$.*

1. *The partition $\mathbf{p}$ has only even parts if and only if $\phi(-I) = -I = \exp(2\pi i H_{n/2})$ and the partition $\mathbf{p}$ has only odd parts if an only if $\phi(-I) = I$.*

2. *If a partition $\mathbf{p}$ has even parts $\mathbf{p_{even}}$ adding up to m and odd parts $\mathbf{p_{odd}}$ adding up to $n-m$, then the corresponding order 2 element s is conjugate to the element with a -1 eigenspace V_{-1} of dimension m and a 1 eigenspace V_1 of dimension $n-m$. It is conjugate to $\exp(\pi i H_{m/2} + H_{n-m/2})$ and has centralizer $G^s = (GL(V_{-1}) \times GL(V_1)) \cap SL(n) = (GL(m) \times GL(m-n)) \cap SL(n)$.*

3. *The partition ϕ is of the form $\phi = \phi_{even} \oplus \phi_{odd}$, where $\phi_{odd} \colon SL(2, \mathbb{C}) \to SL(V_1)$ corresponds to the permutation $\mathbf{p_{odd}}$ as a homomorphism into $SL(m)$ and $\phi_{even} \colon SL(2, \mathbb{C}) \to SL(V_{-1})$ corresponds to the permutation $\mathbf{p_{even}}$ as a homomorphism into $SL(n-m)$.*

Proof. These follow from the representation theory of $SL(2, \mathbb{C})$. The representation factors as

$$\mathbb{C}^n = \sum_{p_i \in \mathbf{p}} V^i,$$

where V_i has dimension p_i and is an irreducible representation of $SL(2, \mathbb{C})$. The eigenspaces of $\phi(-I)$ are

$$V_1 = \bigoplus_{i \colon p_i \text{ odd}} V_i$$

and

$$V_{-1} = \bigoplus_{i:\, p_i \text{ even}} V_i,$$

each of which is a subrepresentation of $\mathbb{C}^n$. Hence, if $\mathbf{p}$ has only even parts, then $\phi(-I) = -I$, and if $\mathbf{p}$ has only odd parts, then $\phi(-I) = I$ and the nilpotent element $d\phi(X_{SL(2)})$ is even. Otherwise, the dimensions of V_1 and V_{-1} imply that $\mathbf{p}$ is conjugate to $\exp(\pi i H_{m/2} + H_{n-m/2})$, where m is the sum of the even parts of $\mathbf{p}$. Let ϕ_{odd} be the restriction of the representation ϕ to V_1 and let ϕ_{even} be the restriction of the representation ϕ to V_1. From the decompositions of V_1 and V_{-1} into irreducibles, it follows that ϕ_{odd} corresponds to the partition $\mathbf{p_{odd}}$ and ϕ_{even} corresponds to the partition $\mathbf{p_{even}}$. From the sum of representations $\mathbb{C} = V_1 \oplus V_{-1}$, it follows that $\phi = \phi_{even} \oplus \phi_{odd}$. $\square$

2.7 Correspondence in $Sp(2n, \mathbb{C})$

In this section, we will discuss the correspondence in Theorem 2.1.7 in the context of the partition classification of nilpotent orbits in $Sp(2n, \mathbb{C})$. In order to do determine the nilpotent element associated to each partition for $Sp(2n, \mathbb{C})$, we will use the standard triples in the Lie algebra as in [4]. The semisimple element of the standard triple will determine the nilpotent orbit associated to the partition.

In $\mathfrak{sp}_{2n}$, nilpotent orbits are in bijection with partitions of 2n, where all odd parts occur an even number of times. The Lie algebra $\mathfrak{sp}_{2n}$ may be realized as the set of matrices:

$$\left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^t \end{pmatrix} : A_i \; n \times n \text{ matrix}, A_2, A_3 symmetric \right\}.$$

A maximal torus $\mathfrak{t}$ is given by the diagonal matrices in $\mathfrak{sp}_{2n}$. Let e_i be the linear functional on $\mathfrak{t}$ picking out the nth entry. The roots of $\mathfrak{sp}_{2n}$ are $\{\pm e_i \pm e_j, 2e_k : 1 \leq i, j \leq n, 1 \leq k \leq n\}$. A set of simple roots with the same numbering as the standard Dynkin diagram is given by

$$\alpha_i = \begin{cases} e_i - e_{i+1} & \text{if } 1 \le i \le n-1 \\ 2e_n & \text{if } i = n \end{cases}$$

xc Let E_{ij} be the matrix with one in the (i,j) entry and zero everywhere else.

To attach the semisimple element $H_{\mathbf{p}}$ of a standard triple to a partition $\mathbf{p}$, we break the partition into chunks which are each either pairs of equal odd parts, or single even parts. To each even even chunk of size q, assign consecutive indices $\{j+1,\dots,j+q\}$ and for an odd chunk with a pair of equal parts of size $2r+1$, assign consecutive indices $\{l+1,\dots,l+2r+1\}$ so that the the indices assigned to different chunks are disjoint. Any such choice of chunks will lead to the same orbit. Then, for each even chunk $\mathcal{C}$, let

$$H_{\mathcal{C}} = \sum_{l=1}^{q}(2q-2l+1)(E_{j+l,j+l} - E_{n+j+l,n+j+l})$$

and for an odd chunk $\mathcal{C}$ with two parts of size r,

$$H_{\mathcal{C}} = \sum_{m=0}^{2r}(2r-2m)(E_{l+1+m,l+1+m} - E_{n+l+1+m,n+l+1+m}).$$

Adding up all of the elements $H_{\mathcal{C}}$ gives the semisimple element $H_{\mathbf{p}}$ in the standard triple. Since we have not specified which indices are attached to each chunk, we have only specified $H_{\mathbf{p}}$ up to conjugacy.

Using these standard triples and the following proposition, we can compute the order 2 element associated to each nilpotent orbit.

Proposition 2.7.1. *Let $\phi\colon SL(2,\mathbb{C}) \to Sp(2n,\mathbb{C})$ be a homomorphism corresponding to a partition $\mathbf{p}$.*

1. *The partition $\mathbf{p}$ has only odd parts if and only if $\phi(-I) = I$ and $\mathbf{p}$ has only even parts if and only if $\phi(-I) = -I = \exp(2\pi i H_n)$.*

2. *If $\mathbf{p}$ has even parts adding to $2m$ and odd parts adding to $2n-2m$, then $\phi(-I)$ is conjugate to $\exp(\pi i H_m)$, which has centralizer $Sp(2m) \times Sp(2n-2m)$.*

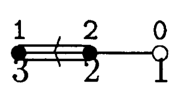
Proof. Since $Sp(2n, \mathbb{C})$ embeds in $SL(n, \mathbb{C})$, the matrix exponential is equal to the group exponential. From Proposition 2.2.1, $\exp(2\pi i H_n)$ is the only nontrivial central element of $Sp(2n, \mathbb{C})$ and so $\exp(2\pi i H_n) = -I$.

As in $SL(n, \mathbb{C})$, the partition $\mathbf{p}$ has only even (resp. odd) parts if and only if all diagonal entries of the diagonal matrix $H_{\mathbf{p}}$ are odd (resp. even), so that $\exp(\pi i H_{\mathbf{p}}) = -I$ (resp. $\exp(\pi i H) = I$) if and only if the partition has only even (resp. odd) parts.

For the second part, assign indices to the chunks so that the first m indices are assigned to the chunks containing a pair of equal odd parts and the last $n - m$ indices are assigned to the chunks containing even parts. Then the diagonal matrix $\exp(\pi i H_{\mathbf{p}})$ has first m diagonal entries equal to 1 and the next $n - m$ diagonal entries equal to -1, the next $n - m$ diagonal entries equal to 1 and the next m diagonal entries equal to -1, and the final $n - m$ entries equal to 1. The fundamental coweight H_i (for $1 \leq i \leq n - 1$) is a diagonal matrix with first $n - m$ diagonal entries equal to 1, next m diagonal entries equal to 0, next $n - m$ diagonal entries equal to -1 and last m diagonal entries equal to 0. For this ordering $\exp(\pi i H_i) = \exp(\pi i H_{\mathbf{p}})$, and from Theorem 2.5.1, the centralizer $G^{\exp(\pi i H_{\mathbf{p}})}$ is equal to $Sp(2m) \times Sp(2n - 2m)$. □

2.8 Correspondence in G_2

We will discuss nilpotent orbits in type G_2 in order to illustrate how the techniques we have discussed allow a computation of the bijection in Theorem 3.1.13. Let G be the connected simply connected, complex group with Lie algebra G_2. Denote the extended diagram



The even nilpotent orbits in G_2 can be found in [4], and have the following weighted Dynkin diagrams, where we let H_ϕ be the semimsimple element in the corresponding standard triple.

$\alpha_1(H_\phi)$	$\alpha_2(H_\phi)$
0	0
0	2
2	2

We may apply the algorithm in Proposition 2.3.4 to find an element H_s so that $\exp(\pi i H_s)$ is conjugate to $\exp(\pi i H_\phi)$ and in the fundamental alcove. Since the center of G_2 is trivial, $H_s = 0$ in all cases.

The non-central order 2 elements in G are all conjugate to $s = \exp(\pi i H_2)$, which has centralizer G^s of type $A_1 \times A_1$ where the simple roots of G^s can be chosen to be α_0 and α_1 by Theorem 2.5.1. Below we have listed all of the Dynkin diagrams for even nilpotent orbits in G^s.

$\alpha_0(H'_\phi)$	$\alpha_2(H'_\phi)$
0	0
2	0
0	2
2	2

The simply connected cover $\widetilde{G^s}$ of G^s is isomorphic to $SL(2) \times SL(2)$. Let $H'_s \in \mathfrak{t}$ be the element so that $H'_s/2$ is in the fundamental alcove and conjugate to $H'_\phi/2$ by the affine Weyl group of $\widetilde{G_s}$. An element H'_ϕ corresponds to a nilpotent orbit in G_2 by the correspondence in Theorem 2.1.7 if and only if $H'_s/2$ is conjugate to L_0 or L_2 by the affine Weyl group of $\widetilde{G^s}$. From our discussion of nilpotent orbits in $SL(n, \mathbb{C})$, from Proposition 2.3.4, or from recognizing that in all cases $H'_\phi/2$ already is in the fundamental alcove, we can identify which order 2 elements associated to each of the nilpotent orbits in $A_1 \times A_1$. These elements H'_s are listed in the table below.

$\alpha_1(H'_\phi)$	$\alpha_0(H'_\phi)$	$\alpha_1(H'_s)$	$\alpha_0(H'_s)$
0	0	0	0
2	0	2	0
0	2	0	2
2	2	2	2

For the elements H'_ϕ where $H'_s = 2L_0$ or $H'_s = 2L_2$, we find the value $\alpha_1(H'_\phi)$ from the equation $\alpha_0(H'_\phi) + 3\alpha_1(H'_\phi) + 2\alpha_2(H'_\phi) = 0$.

$\alpha_1(H'_\phi)$	$\alpha_2(H'_\phi)$	$\alpha_0(H'_\phi)$
2	-3	0
0	-1	2

Finally, we apply Lemma 2.3.3 to find an element $H_\phi \in \mathfrak{t}$, which is conjugate to H'_ϕ by the Weyl group of G and dominant with respect to Π, the simple roots for (G, T). The results are listed in Table 2.5.

Table 2.5:

$\alpha_1(H'_\phi)$	$\alpha_2(H'_\phi)$	$\alpha_0(H'_\phi$	$\alpha_1(H_\phi)$	$\alpha_2(H_\phi)$	$\alpha_0(H_\phi)$
2	-3	0	1	0	-3
0	-1	2	0	1	-2

This produces the form of the bijection in Theorem 2.1.7 on Dynkin diagrams. The same algorithm can be followed for any group to find the correspondence explicitly. In section 2.9, we list the bijection for the remaining exceptional groups.

2.9 Correspondence for Exceptional Groups

We have included the correspondence in Theorem 2.1.7 below. For this, we use the following notations. Let G be an (exceptional) group, let T be a torus in G and let Π be a set of simple roots for (G, T). The element H_ϕ is the dominant semisimple element in $\mathfrak{t}$ which is in a standard triple for the nilpotent orbit $\mathcal{O}$, so that the elements $\alpha_i(H_\phi)$ are the coefficients in the Dynkin diagram. The element H_s is an element of $\mathfrak{t}$ for which $H_s/2$ is in the fundamental alcove and $\exp(\pi i H_\phi)$ is conjugate to $\exp(\pi i H_s)$. Let $s = \exp(2\pi i H_s)$. Let Π_s be the set of simple roots for G^s given in Theorem 2.5.1. Let $\widetilde{G^s}$ be the universal cover of $(G^s)'$ and

$$\sigma\colon \widetilde{G^s} \to G^s$$

be the projection. Let $\phi\colon SL(2,\mathbb{C}) \to G^s$ be a homomorphism so that $\sigma(\phi(\pi i H_{SL(2)})) = \exp(\pi i H_s)$. Then H'_ϕ is the dominant element of $\mathfrak{t}$ conjugate to $d\phi(H_{SL(2)})$ by $\widetilde{G^s}$. We list below the values of $\beta_i(H'_\phi)$, where β_i are the roots of $\mathfrak{g}^s$ as numbered as in the diagram. These are coefficients of the weighted Dynkin diagram of the corresponding nilpotent orbit in $\mathfrak{g}^y$. The element $H_s \in \mathfrak{t}$ is the element so that $H_s/2$ is in the fundamental alcove for $(G^s)'$ with respect to the simple roots Π_s and $\exp(\pi i H'_s)$ is conjugate to $\exp(\pi i H'_\phi)$.

Table 2.6:

Nilpotent Orbits in type G_2							
G				G^s			
 α_1 α_2				 β_1 β_2			
H_ϕ		H_s		H'_ϕ		H'_s	
0	0	0	0				
0	1	0	1	0	2	0	2
1	0	0	1	2	0	2	0
2	0	0	0				
2	2	0	0				

Table 2.7:

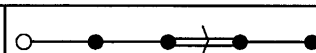
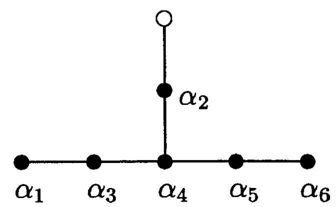
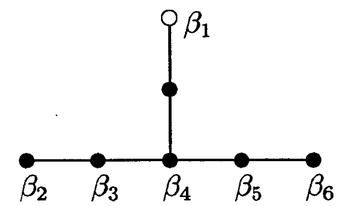
Nilpotent Orbits in Type F_4					
G		G^s			
 α_1 α_2 α_3 α_4		 β_1 β_2 β_3 β_4		 β_1 β_2 β_3 β_4	
H_ϕ	H_s	H'_ϕ	H'_s	H'_ϕ	H'_s
0 0 0 0	0 0 0 0				
1 0 0 0	1 0 0 0	2 0 0 0	2 0 0 0		
0 0 0 1	0 0 0 1			2 0 0 0	2 0 0 0
0 1 0 0	1 0 0 0	0 2 0 0	0 2 0 0		
2 0 0 0	0 0 0 0				
0 0 0 2	0 0 0 0				
0 0 1 0	0 0 0 1			0 0 2 0	2 0 0 0
2 0 0 1	0 0 0 1			2 2 0 0	2 0 0 0
0 1 0 1	1 0 0 0	2 0 2 0	2 0 0 0		
1 0 1 0	1 0 0 0	0 2 0 2	0 2 0 0		
0 2 0 0	0 0 0 0				
2 2 0 0	0 0 0 0				
1 0 1 2	1 0 0 0	0 2 2 2	0 2 0 0		
0 2 0 2	0 0 0 0				
2 2 2 2	0 0 0 0				

Table 2.8:

Nilpotent orbits in type E_6 with $H_s = H_2$

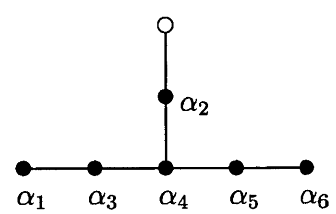


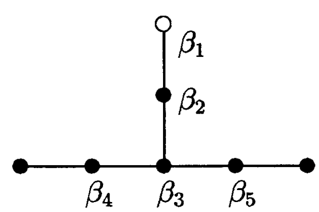


H_ϕ						H_s						H'_ϕ						H'_s					
0	1	0	0	0	0	0	1	0	0	0	0	2	0	0	0	0	0	2	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	2	0	0	0	0	0	2	0	0
1	1	0	0	0	1	0	1	0	0	0	0	2	2	0	0	0	2	2	0	0	0	0	0
1	0	0	1	0	1	0	1	0	0	0	0	2	0	2	0	2	0	2	0	0	0	0	0
0	1	1	0	1	0	0	1	0	0	0	0	0	2	0	2	0	2	0	0	0	2	0	0
1	1	1	0	1	1	0	1	0	0	0	0	2	2	2	0	2	2	2	0	0	0	0	0
2	1	1	0	1	2	0	1	0	0	0	0	0	2	2	2	2	2	0	0	0	2	0	0

Table 2.9:

Nilpotent Orbits in type E_6 with $H_s = H_1 + H_5$

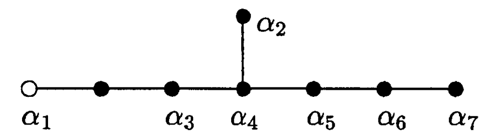




H_ϕ						H_s						H'_ϕ					H'_s				
1	0	0	0	0	1	1	0	0	0	0	1	2	0	0	0	0	2	0	0	0	0
0	0	1	0	1	0	1	0	0	0	0	1	0	0	2	0	0	2	0	0	0	0
1	2	0	0	0	1	1	0	0	0	0	1	2	2	0	0	0	2	0	0	0	0
1	2	1	0	1	1	1	0	0	0	0	1	2	2	0	2	2	2	0	0	0	0

Table 2.10:

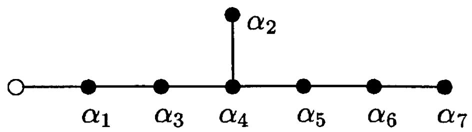
Even Nilpotent Orbits in E_7

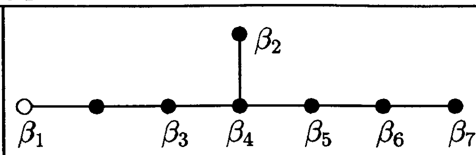


H_ϕ							H_s						
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	2	0	0	0	0	0	0	2
2	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	2	0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0	0	0	0	0	0	2
2	0	0	0	0	0	2	0	0	0	0	0	0	2
0	0	2	0	0	0	0	0	0	0	0	0	0	0
2	0	2	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	2	0	0	0	0	0	0	0	0
0	0	0	0	2	0	0	0	0	0	0	0	0	2
2	0	0	0	0	2	2	0	0	0	0	0	0	2
0	0	0	2	0	0	0	0	0	0	0	0	0	0
2	0	0	0	2	0	0	0	0	0	0	0	0	2
0	0	2	0	0	2	0	0	0	0	0	0	0	0
2	0	2	0	0	2	0	0	0	0	0	0	0	0
0	0	0	2	0	0	2	0	0	0	0	0	0	2
0	0	0	2	0	2	0	0	0	0	0	0	0	0
2	0	0	2	0	0	2	0	0	0	0	0	0	2
2	0	0	2	0	2	0	0	0	0	0	0	0	0
2	0	2	2	0	2	0	0	0	0	0	0	0	0
2	0	0	2	0	2	2	0	0	0	0	0	0	2
2	2	2	0	2	0	2	0	0	0	0	0	0	2
2	2	2	0	2	2	2	0	0	0	0	0	0	2
2	2	2	2	2	2	2	0	0	0	0	0	0	2

Table 2.11:

Nilpotent orbits for type E_7 with $H_s = H_6$

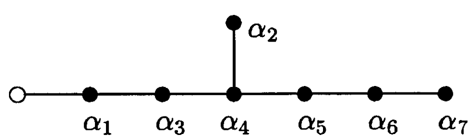


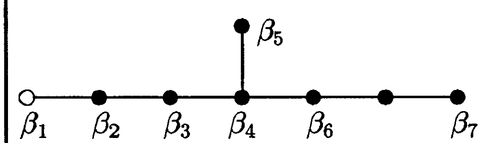


H_ϕ							H_s	H'_ϕ							H'_s
1	0	0	0	0	0	0	1 0 0 0 0 0 0	2	0	0	0	0	0	0	2 0 0 0 0 0 0
0	0	1	0	0	0	0	1 0 0 0 0 0 0	0	0	2	0	0	0	0	0 0 2 0 0 0 0
1	0	0	0	0	1	0	1 0 0 0 0 0 0	2	0	0	0	0	2	0	2 0 0 0 0 0 0
0	0	1	0	0	1	0	1 0 0 0 0 0 0	2	0	0	2	0	0	0	2 0 0 0 0 0 0
1	0	0	1	0	0	0	1 0 0 0 0 0 0	0	0	2	0	0	2	0	0 0 2 0 0 0 0
0	1	1	0	0	0	1	1 0 0 0 0 0 0	2	0	0	0	2	0	2	2 0 0 0 0 0 0
2	1	1	0	0	0	1	1 0 0 0 0 0 0	2	0	0	0	2	2	2	2 0 0 0 0 0 0
1	0	0	1	0	1	0	1 0 0 0 0 0 0	2	0	0	2	0	2	0	2 0 0 0 0 0 0
1	0	0	1	0	2	0	1 0 0 0 0 0 0	0	0	2	2	0	2	0	0 0 2 0 0 0 0
2	1	1	0	1	1	0	1 0 0 0 0 0 0	2	0	0	2	2	2	2	2 0 0 0 0 0 0

Table 2.12:

Nilpotent orbits for type E_7 with $H_s = H_1$

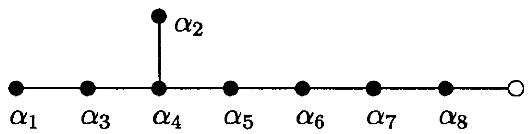


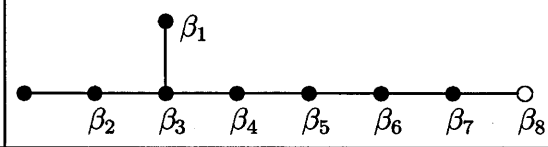


H_ϕ							H_s	H'_ϕ							H'_s
0	0	0	0	0	1	0	0 0 0 0 0 1 0	2	0	0	0	0	0	0	2 0 0 0 0 0 0
0	1	0	0	0	0	1	0 0 0 0 0 1 0	0	0	0	0	2	0	2	0 0 0 0 2 0 2
0	0	0	1	0	0	0	0 0 0 0 0 1 0	0	0	0	2	0	0	0	2 0 0 0 0 0 0
2	0	0	0	0	1	0	0 0 0 0 0 1 0	2	2	0	0	0	0	0	2 0 0 0 0 0 0
1	0	0	0	1	0	1	0 0 0 0 0 1 0	0	2	0	0	2	0	2	0 0 0 0 2 0 2
0	0	0	1	0	1	0	0 0 0 0 0 1 0	2	0	0	2	0	0	0	2 0 0 0 0 0 0
2	0	0	1	0	1	0	0 0 0 0 0 1 0	2	2	0	2	0	0	0	2 0 0 0 0 0 0
1	0	0	1	0	1	2	0 0 0 0 0 1 0	0	2	0	2	2	0	2	0 0 0 0 2 0 2
0	1	1	0	1	0	2	0 0 0 0 0 1 0	2	0	2	0	2	2	0	2 0 0 0 0 0 0
2	1	1	0	1	0	2	0 0 0 0 0 1 0	2	2	2	0	2	2	0	2 0 0 0 0 0 0
2	1	1	0	1	2	2	0 0 0 0 0 1 0	2	2	2	2	2	2	0	2 0 0 0 0 0 0

Table 2.13:

Nilpotent Orbits in type E_8 with $H_s = H_1$

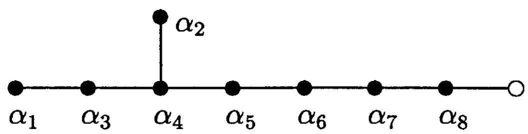


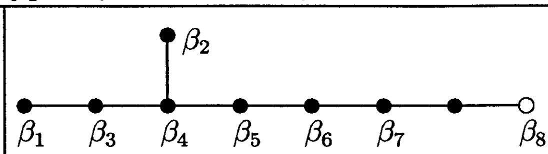


H_ϕ								H_s	H'_ϕ								H'_s
1	0	0	0	0	0	0	0	10000000	0	0	0	0	0	0	0	2	20000000
0	1	0	0	0	0	0	0	10000000	2	0	0	0	0	0	0	0	02000000
0	0	0	0	0	1	0	0	10000000	0	0	0	0	0	2	0	0	20000000
1	0	0	0	0	0	0	2	10000000	0	0	0	0	0	0	2	2	20000000
0	0	0	0	1	0	0	0	10000000	0	0	0	2	0	0	0	0	20000000
0	0	1	0	0	0	0	1	10000000	2	0	0	0	0	0	2	0	02000000
1	0	0	0	0	1	0	0	10000000	0	0	0	0	2	0	0	2	20000000
1	0	0	0	1	0	0	0	10000000	2	0	0	0	2	0	0	0	02000000
1	0	0	0	0	1	0	2	10000000	0	0	0	0	2	0	2	2	20000000
0	0	0	1	0	0	0	1	10000000	0	0	0	2	0	0	2	0	20000000
0	0	0	1	0	0	1	0	10000000	0	0	2	0	0	2	0	0	20000000
1	0	0	1	0	0	0	1	10000000	2	0	0	0	2	0	2	0	02000000
0	0	1	0	0	1	0	1	10000000	0	0	2	0	0	0	2	2	20000000
0	1	1	0	0	0	1	0	10000000	0	0	0	2	0	2	0	2	20000000
0	1	1	0	0	0	1	2	10000000	0	0	0	2	0	2	2	2	20000000
2	1	1	0	0	0	1	2	10000000	0	0	0	2	2	2	2	2	20000000
1	0	0	1	0	1	0	1	10000000	0	0	2	0	2	0	2	2	20000000
1	0	0	1	0	1	1	0	10000000	2	0	2	0	2	0	2	0	02000000
2	1	1	0	1	1	0	1	10000000	0	0	2	2	2	2	2	2	20000000

Table 2.14:

Nilpotent Orbits in type E_8 with $H_s = H_8$





H_ϕ								H_s	H'_ϕ								H'_s
0	0	0	0	0	0	0	1	0 0 0 0 0 0 0 1	0	0	0	0	0	0	0	2	0 0 0 0 0 0 0 2
0	0	0	0	0	0	1	0	0 0 0 0 0 0 0 1	0	0	0	0	0	0	2	0	0 0 0 0 0 2 0 0
1	0	0	0	0	0	0	1	0 0 0 0 0 0 0 1	2	0	0	0	0	0	0	2	0 0 0 0 0 0 0 2
0	0	1	0	0	0	0	0	0 0 0 0 0 0 0 1	0	2	0	0	0	0	0	0	0 0 0 0 0 2 0 0
1	0	0	0	0	0	1	0	0 0 0 0 0 0 0 1	0	0	0	0	0	2	0	2	0 0 0 0 0 0 0 2
0	0	0	0	0	1	0	1	0 0 0 0 0 0 0 1	2	0	0	0	0	0	2	0	0 0 0 0 0 2 0 0
0	1	0	0	0	0	1	0	0 0 0 0 0 0 0 1	0	0	2	0	0	0	0	2	0 0 0 0 0 0 0 2
0	0	0	1	0	0	0	0	0 0 0 0 0 0 0 1	0	0	0	0	2	0	0	0	0 0 0 0 0 2 0 0
0	1	0	0	0	0	1	2	0 0 0 0 0 0 0 1	2	0	2	0	0	0	0	2	0 0 0 0 0 0 0 2
1	0	0	0	0	1	0	1	0 0 0 0 0 0 0 1	2	0	0	0	0	2	0	2	0 0 0 0 0 0 0 2
2	0	0	0	0	1	0	1	0 0 0 0 0 0 0 1	2	0	0	0	0	2	2	0	0 0 0 0 0 0 0 2
0	0	0	1	0	0	0	2	0 0 0 0 0 0 0 1	2	0	0	0	2	0	0	0	0 0 0 0 0 2 0 0
0	0	1	0	0	1	0	0	0 0 0 0 0 0 0 1	0	0	0	2	0	0	0	2	0 0 0 0 0 0 0 2
1	0	0	0	1	0	1	0	0 0 0 0 0 0 0 1	0	0	2	0	0	2	0	2	0 0 0 0 0 0 0 2
0	0	0	1	0	1	0	0	0 0 0 0 0 0 0 1	0	0	0	2	0	0	2	0	0 0 0 0 0 2 0 0
1	0	0	0	1	0	1	2	0 0 0 0 0 0 0 1	2	0	2	0	0	2	0	2	0 0 0 0 0 0 0 2
1	0	0	1	0	1	0	0	0 0 0 0 0 0 0 1	0	0	0	2	0	2	0	2	0 0 0 0 0 0 0 2
0	0	0	1	0	1	0	2	0 0 0 0 0 0 0 1	2	0	0	2	0	0	2	0	0 0 0 0 0 2 0 0
1	0	0	1	0	1	0	2	0 0 0 0 0 0 0 1	2	0	0	2	0	2	0	2	0 0 0 0 0 0 0 2
2	0	0	1	0	1	0	2	0 0 0 0 0 0 0 1	2	0	0	2	0	2	2	0	0 0 0 0 0 2 0 0
1	0	0	1	0	1	2	2	0 0 0 0 0 0 0 1	2	0	2	2	0	2	0	2	0 0 0 0 0 0 0 2
0	1	1	0	1	0	2	2	0 0 0 0 0 0 0 1	2	2	2	0	2	0	2	0	0 0 0 0 0 2 0 0
2	1	1	0	1	0	2	2	0 0 0 0 0 0 0 1	2	2	2	0	2	2	2	0	0 0 0 0 0 2 0 0
2	1	1	0	1	2	2	2	0 0 0 0 0 0 0 1	2	2	2	2	2	2	2	0	0 0 0 0 0 2 0 0

Chapter 3

Real Groups

3.1 Proof of Nilpotent Orbit Correspondence for Real Groups

Let G be a complex reductive, simply connected, connected Lie group. Let T be a maximal torus of G and let $(R, X, R^\vee, X^\vee)$ be a corresponding root datum and let Π be a set of simple roots for G.

Definition 3.1.1.

1. A real form of G is an antiholomorphic Lie group involution σ of G. For a real form σ, let
$$G(\mathbb{R}, \sigma) = G^\sigma$$
and let $\mathfrak{g}(\mathbb{R}, \sigma)$ be the Lie algebra of $G(\mathbb{R}, \sigma)$. A compact real form is a real form σ such that G^σ is compact. An equivalence class of real forms is a G conjugacy class of real forms of G.

2. A Cartan involution θ for a real form σ is a holomorphic involution such that $\sigma\theta$ is a compact real form. For a holomorphic involution θ, let
$$K_\theta = G^\theta$$

and let $\mathfrak{p}_\theta$ be the -1-eigenspace of θ on $\mathfrak{g}$. If θ is a Cartan involution for a real form σ, let

$$K(\mathbb{R}, \sigma, \theta) = (G^\sigma)^\theta.$$

An equivalence class of holomorphic involutions is a G-conjugacy class of holomorphic involutions of G.

Cartan's theorem below describes the relation between real forms of G and holomorphic involutions of G.

Theorem 3.1.2. *([11], Proposition 1.143)*

1. *Let σ_0 be a real form of G. Then there is a compact real form σ_c of G that commutes with σ_0. This compact real form is unique up to conjugation by G^{σ_0}. The composition*

$$\theta = \sigma_0 \circ \sigma_c$$

is a Cartan involution for the real form σ_0.

2. *Suppose θ is an holomorphic involution of G. Then there is a compact real form σ_c of G that commutes with θ. This compact real form is unique up to conjugation by K_θ. The composition*

$$\sigma_0 = \theta \circ \sigma_c$$

is a real form of G.

3. *The constructions above define a bijection between equivalence classes of real forms of G and equivalence classes of holomorphic involutions of G.*

Definition 3.1.3. 1. Let σ be a real form. A nilpotent element for σ is a nilpotent element of $\mathfrak{g}$ in $\mathfrak{g}(\mathbb{R}, \sigma)$. A nilpotent orbit for the real form σ is a $G(\mathbb{R}, \sigma)$-conjugacy class of nilpotent elements for σ.

2. Let θ be a holomorphic involution. A nilpotent element for θ is a nilpotent element in $\mathfrak{p}_\theta$. A nilpotent orbit for the holomorphic involution θ is a K_θ-conjugacy class of nilpotent elements for θ.

Let

$$x_{SL(2)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The involution $\theta_{SL(2)} = \mathrm{Ad}(x_{SL(2)})$ acts as a Cartan involution for the real form

$$\sigma_{SL(2)} : SL(2,\mathbb{C}) \to SL(2,\mathbb{C}), g \mapsto \overline{g}$$

with $G^{\sigma_{SL(2)}} = SL(2,\mathbb{R})$. Let

$$X_{SL(2),\theta} = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C}).$$

Definition 3.1.4. 1. Let σ be an antiholomorphic involution of G. An $SL(2)$-homomorphism for σ is a homomorphism

$$\phi\colon SL(2,\mathbb{C}) \to G$$

such that

$$\sigma \circ \phi = \phi \circ \sigma_{SL(2)}$$

.

2. Let θ be a holomorphic involution of G. An $SL(2)$-homomorphism for θ is a homomorphism

$$\phi\colon SL(2,\mathbb{C}) \to G$$

such that

$$\theta \circ \phi = \phi \circ \theta_{SL(2)}.$$

The Kostant-Sekiguchi bijection below relates nilpotent orbits for real forms and nilpotent orbits for holomorphic involutions.

Theorem 3.1.5. *([4], Theorem 9.5.1) Let θ be a Cartan involution for the real form σ.*

Let X be a nilpotent element for the real form σ. Then there is a homomorphism ϕ for σ such that

$$d\phi(X_{SL(2)}) = X.$$

Then ϕ is conjugate by an element of $G(\mathbb{R}, \sigma)$ to an $SL(2)$-homomorphism ϕ' for θ and σ. The homomorphism ϕ' is determined up to conjugacy by the group $K(\mathbb{R}, \sigma, \theta)$. The element $d\phi'(X_{SL(2),\theta})$ is a nilpotent element for θ.

Let X be a nilpotent element for the holomorphic involution θ. Then there is a homomorphism ϕ for σ such that

$$d\phi(X_{SL(2)}, \theta) = X.$$

Then ϕ is conjugate by an element of K_θ to an $SL(2)$-homomorphism ϕ' for θ and σ. The homomorphism ϕ' is determined up to conjugacy by the group $K(\mathbb{R}, \sigma, \theta)$. The element $d\phi'(X_{SL(2)})$ is a nilpotent element for σ.

The constructions above define a bijection between nilpotent orbits for the real form σ and nilpotent orbits for the holomorphic involution θ.

Definition 3.1.6. 1. An inner class of automorphisms of G is the set

$$\{\mathrm{Ad}(x)\gamma : x \in G\}$$

for a fixed automorphism γ of G of order two.

2. Let γ be a fixed automorphism of G. Let

$$G^{\Gamma} = G \rtimes \{1, \delta\}$$

where $\delta g \delta^{-1} = \gamma(g)$ and $\delta^2 = 1$.

The group G^{Γ} has two connected components, one of which is $G \times \{1\}$, which we will refer to as G.

3. A strong involution of (G, γ) is an element $x \in G^{\Gamma} \backslash G$ such that $x^2 \in \mathcal{Z}(G)$. Let $\theta_x = Ad(x)$, let $K_x = G^x$, and let $\mathfrak{p}_x$ be the -1 eigenspace of θ_x. We define an equivalence class of strong involutions to be an orbit of G under the conjugation action.

   A G-orbit of strong involutions is an equivalence class of strong involutions.

The map

$$x\delta \mapsto \mathrm{Ad}(x)\gamma$$

takes strong involutions of (G, γ) onto holomorphic involutions of G in the inner class of γ. For any holomorphic involution $\theta = \mathrm{Ad}(x)\gamma$ in the inner class of γ, the pre-image of θ under this map is $x\mathcal{Z}(G)\delta$. This map induces a map of equivalence classes of strong involutions of (G, γ) to equivalence classes of holomorphic involutions for G. For this induced map, the pre-image of an equivalence class of holomorphic involutions $G \cdot \mathrm{Ad}(x)\gamma$ has cardinality at most $|\mathcal{Z}(G)|$, but the cardinality may be smaller.

Definition 3.1.7. A nilpotent element for a strong involution x of (G, γ) is a nilpotent element for $\mathrm{Ad}(x)$. An equivalence class of nilpotent elements for strong involutions of (G, γ) is a G-orbit of pairs (x, X), where X is a nilpotent element for the strong involution x. An equivalence class of nilpotent elements $G \cdot (x, X)$ for strong involutions of (G, γ) is said to be an equivalence class of nilpotent orbits for $G \cdot x$.

Equivalence classes of nilpotent elements for strong involutions of (G,γ) are related to nilpotent orbits for a fixed strong involution by the following proposition.

Proposition 3.1.8. *1. Let $G \cdot x$ be an equivalence class of strong involutions for (G,γ) and let $\mathcal{O}$ be an equivalence class of nilpotent orbits for $G \cdot x$. Then the set*

$$\mathcal{O}' = \{X \in \mathfrak{g} : (x,X) \in \mathcal{O}\} \subset \mathfrak{p}_x$$

is a nilpotent orbit for the holomorphic involution θ_x.

2. The above construction gives a bijection between nilpotent orbits for the equivalence class of strong involutions $G \cdot x$ and nilpotent orbits for the holomorphic involution x.

Proof. For the first part, the set

$$\{X \in \mathfrak{g} : (x,X) \in \mathcal{O}\}$$

is a K_x orbit of nilpotent elements for θ_x, and is hence a nilpotent orbit for θ_x.

For the second part, any nilpotent orbit $K_x \cdot X$ for θ_x is contained in the equivalence class of nilpotent orbits $G \cdot (x,X)$ for the equivalence class of strong involutions $G \cdot x$. The construction in the first part takes $G \cdot (x,X)$ to $K_x \cdot X$. □

Proposition 3.1.9. *The map*

$$(x,\phi) \mapsto (x, d\phi(X_{SL(2),\theta}))$$

induces a bijection between G-orbits of pairs (x,ϕ), where

1. x is a strong involution of (G,γ) and

2. ϕ is an $SL(2)$-homomorphism for the strong involution x, *and equivalence classes of nilpotent orbits for strong involutions of* (G, γ).

Proof. All of the properties enumerated for (x, ϕ) are invariant under the conjugation action of ϕ so G acts on such pairs. The map given is equivariant under the action of G by conjugation, so it induces a map on G orbits.

Next we show that it maps into the G-orbits described. The element $d\phi(X_{SL(2),\theta})$ must be nilpotent as the image of a nilpotent element of a semisimple Lie algebra under a homomorphism of Lie algebras is nilpotent. The condition

$$\mathrm{Ad}(x) \circ \phi = \phi \circ \mathrm{Ad}(x_{SL(2)})$$

implies that

$$\mathrm{Ad}(x) \circ d\phi = d\phi \circ \mathrm{Ad}(x_{SL(2)}).$$

In particular,

$$\begin{aligned} \mathrm{Ad}(x)X &= (\mathrm{Ad}(x) \circ d\phi)(X_{SL(2),\theta}) \\ &= (d\phi \circ \mathrm{Ad}(x_{SL(2)}))X_{SL(2),\theta} \end{aligned}$$

The element $X_{SL(2),\theta}$ is in the -1 eigenspace $\mathrm{Ad}(x_{SL(2)})$, so that

$$\begin{aligned} \mathrm{Ad}(x)X &= -d\phi(X_{SL(2),\theta}) \\ &= -X. \end{aligned}$$

Hence X is a nilpotent element of $\mathfrak{p}_x$ and the mapping $(x, \phi) \mapsto (x, d\phi(X_{SL(2),\theta})$ maps into the given set.

Suppose (x, X) is an equivalence class of nilpotent orbits for strong involutions of (G, γ).

By Theorem 3.1.5, there exists an $SL(2)$-homomorphism ϕ for x satisfying

$$\phi(X_{SL(2),\theta}) = X.$$

Hence the mapping is surjective and the induced mapping is as well.

Let (x_1, ϕ_1) and (x_2, ϕ_2) be two pairs as above with

$$(g \cdot \phi(x_1), g \cdot \phi_1(X_{SL(2),\theta})) = (x_2, \phi_2(X_{SL(2),\theta})).$$

Then $g \cdot \phi_1$ and ϕ_2 satisfy

$$\begin{aligned} \mathrm{Ad}(x) \circ (g \cdot \phi_1) &= (g \cdot \phi_1) \circ \mathrm{Ad}(x_{SL(2)}) \\ \mathrm{Ad}(x) \circ \phi_2 &= \phi_2 \circ \mathrm{Ad}(x_{SL(2)}) \end{aligned}$$

and

$$g \cdot \phi_1(X_{SL(2),\theta}) = \phi_2(X_{SL(2),\theta}) = X.$$

Hence, by Theorem 3.1.5, they are conjugate. Thus, the mapping given is injective. $\square$

The following key lemma will make it possible to parametrize equivalence classes of nilpotent orbits for strong involutions of (G, γ) in a different way.

Lemma 3.1.10. *Let ϕ: $SL(2, \mathbb{C}) \to G$ be a homomorphism and let $x \in G^{\Gamma}$. Then x is a strong involution for (G, γ) and ϕ is an $SL(2)$-homomorphism for the strong involution x if and only if the element*

$$y = x\phi(x_{SL(2)})$$

satisfies the properties

1. $y \in G^\Gamma \backslash G$;

2. $\mathrm{Ad}(y) \circ \phi = \phi$; *and*

3. $y^2 \phi(-I) \in \mathcal{Z}(G)$.

Proof. First, suppose x is a strong involution for (G, Γ) and ϕ is an $SL(2)$-homomorphism for x. Let

$$y = x\phi(x_{SL(2)}).$$

Since

$$\phi(x_{SL(2)}) \in G$$

and

$$x \in G^\Gamma \backslash G,$$

it follows that

$$x\phi(x_{SL(2)}) \in G^\Gamma \backslash G.$$

Then

$$\begin{aligned} \mathrm{Ad}(y) \circ \phi &= \mathrm{Ad}(x\phi(x_{SL(2)}) \circ \phi \\ &= \mathrm{Ad}(x) \circ \phi \circ \mathrm{Ad}(x_{SL(2)})) \\ &= \mathrm{Ad}(x) \circ \mathrm{Ad}(x) \circ \phi \\ &= \mathrm{Ad}(x^2) \circ \phi. \end{aligned}$$

Since $x^2 \in \mathcal{Z}(G)$, it follows that

$$\mathrm{Ad}(x^2) = id_G$$

and so

$$\mathrm{Ad}(y) \circ \phi = \phi.$$

In particular this implies that

$$x\phi(x_{SL(2)}) = \phi(x_{SL(2)})x$$

and so

$$\begin{aligned} y^2 &= x\phi(x_{SL(2)})x\phi(x_{SL(2)}) \\ &= x^2\phi(x_{SL(2)})^2. \end{aligned}$$

Since

$$x_{SL(2)}^2 = -I,$$

we find that

$$y^2 = x^2\phi(-I).$$

Since

$$x^2 \in \mathcal{Z}(G),$$

we find that

$$y^2\phi(-I) \in \mathcal{Z}(G).$$

Thus y obeys all three properties described in the proposition.

Now suppose that

$$y = x\phi(x_{SL(2)}) \in G^\Gamma \backslash G,$$

with

$$\mathrm{Ad}(y) \circ \phi = \phi$$

and

$$y^2\phi(-I) \in \mathcal{Z}(G).$$

Then since $\phi(x_{SL(2)}) \in G$, but $y \in G^\Gamma \backslash G$, it follows that $x \notin G$. Therefore

$$\begin{aligned}
\mathrm{Ad}(x) \circ \phi &= \mathrm{Ad}(y) \circ \mathrm{Ad}(x_{SL(2)}^{-1}) \circ \phi \\
&= \mathrm{Ad}(y) \circ \phi \circ \mathrm{Ad}(x_{SL(2)}^{-1}).
\end{aligned}$$

Since $\mathrm{Ad}(y) \circ \phi = \phi$, it follows that

$$\mathrm{Ad}(x) \circ \phi = \phi \circ \mathrm{Ad}(x_{SL(2)}^{-1}).$$

As the Cartan involution on $SL(2, \mathbb{C})$ given by $\mathrm{Ad}(x_{SL(2)})$ is an involution, it follows that

$$\mathrm{Ad}(x) \circ \phi = \phi \circ \mathrm{Ad}(x_{SL(2)})$$

As before, the fact that $\mathrm{Ad}(y) \circ \phi = \phi$ implies that y commutes with $\phi(x_{SL(2)})$ and so

$$x^2 = y^2\phi(-I).$$

Since $y^2 \in \mathcal{Z}(G)\phi(-I)$ and $\phi(-I)^2 = \phi(I) = I$, it follows that

$$x^2 \in \mathcal{Z}(G).$$

Thus x is a strong involution for (G, Γ) and ϕ is an $SL(2)$-homomorphism for x. $\square$

Using Lemma 3.1.10, we can replace the datum x in the bijection of Proposition 3.1.9 with the datum y. Instead of the condition $\mathrm{Ad}(y) \circ \phi = \phi$, we will use the equivalent condition that ϕ maps into G^y.

Proposition 3.1.11. *The set of equivalence classes of nilpotent orbits for strong involutions of (G, γ) is in bijection with pairs*

$$(y, \phi) \in \left(G^{\Gamma} \backslash G\right) \times \mathrm{Hom}(SL(2, \mathbb{C}), G^y)$$

modulo conjugation by G, where

$$y^2 \phi(-I) \in \mathcal{Z}(G).$$

The equivalence class of nilpotent orbits for strong involutions corresponding to the pair (y, ϕ) is the G-orbit of the pair $(y\phi(x_{SL(2)}), \phi(X_{SL(2),\theta})$

This proposition points to an invariant of each nilpotent orbit for a strong involution of (G, γ): the conjugacy class of the element y associated to it by this proposition. Note that y must be an element of $G^{\Gamma} \backslash G$ whose fourth power is in $\mathcal{Z}(G)$. The set of equivalence classes of nilpotent orbits for strong involutions of (G, γ) has a description in terms of even nilpotent orbits of complex reductive Lie algebras as we will show below.

Lemma 3.1.12. *The map*

$$(y, \phi) \mapsto (y, d\phi(X_{SL(2),\theta})$$

induces a bijection between pairs $(y, \phi) \in G^{\Gamma} \times \mathrm{Hom}(SL(2, \mathbb{C}), G^y)$ modulo conjugation by G, where

$$y^2 \phi(-I) \in \mathcal{Z}(G),$$

and pairs $(y, X) \in G^{\Gamma} \times \mathfrak{g}^y$ modulo conjugation by G, where

1. *X is an even nilpotent element of $\mathfrak{g}^y$ and*

2. *Whenever $\phi' \colon SL(2, \mathbb{C}) \to G^y$ is a homomorphism satisfying $d\phi'(X_{SL(2),\theta}) = X$, we have $y^2 \phi'(-I) \in \mathcal{Z}(G)$.*

Proof. The map $(y, \phi) \mapsto (y, d\phi(X_{SL(2),\theta})$ is G-equivariant so the induced map takes G-orbits to G-orbits.

Let (y, ϕ) be as in the proposition. The element $d\phi(X_{SL(2),\theta})$ must be nilpotent since $X_{SL(2),\theta}$ is nilpotent and $d\phi$ is a homomorphism from a semisimple Lie algebra to a reductive Lie algebra.

Let $\phi'\colon SL(2, \mathbb{C}) \to G^y$ be another homomorphism satisfying $d\phi'(X_{SL(2),\theta}) = d\phi(X_{SL(2),\theta})$. Then by Theorem 2.1.1, it follows that ϕ' is conjugate to ϕ in G^y. Since $y^2\phi(-I) \in \mathcal{Z}(G)$ and $y^2 \in \mathcal{Z}(G^y)$, it follows that $\phi(-I) \in \mathcal{Z}(G^y)$. Since ϕ' and ϕ are conjugate in G^y, $\phi(-I)$ and $\phi'(-I)$ are as well. As $\phi(-I) \in \mathcal{Z}(G^y)$ and $\phi'(-I)$ is conjugate to $\phi(-I)$ in G^y, it follows that $\phi'(-I) = \phi(-I)$. Hence $y^2\phi'(-I) \in \mathcal{Z}(G)$. Since $\phi(-I) \in y^{-2}\mathcal{Z}(G)$ and $y^{-2} \in \mathcal{Z}(G^y)$, it follows that $\phi(-I) \in \mathcal{Z}(G^y)$. Then, by Proposition 2.1.4, the element X is an even nilpotent element of G^y. Thus, the mapping maps into the set described above.

Let (y, X) be a pair as in the theorem. Then, by Proposition 2.1.3, it follows that there exists a homomorphism $\phi\colon SL(2, \mathbb{C}) \to G^y$ so that $d\phi(X_{SL(2),\theta}) = X$. From the conditions on (y, X), it follows that $y^2\phi(-I) \in \mathcal{Z}(G)$. Hence the mapping is surjective, so the induced mapping must be surjective as well.

Let (y_1, ϕ_1) and (y_2, ϕ_2) be a pair so that $g \cdot (y_2, \phi_2) = (y_1, \phi_1)$. Then $\phi_1, g \cdot \phi_2 \in \mathrm{Hom}(SL(2, \mathbb{C}), G^{y_1})$ and $g \cdot \phi_2(X_{SL(2),\theta}) = \phi_1(X_{SL(2),\theta})$. By Theorem 2.1.1, it follows that there is an element $h \in G^{y_1}$ so that $hg \cdot \phi_2$ is conjugate to ϕ_1. Then

$$\begin{aligned} hg \cdot (y_2, \phi_2) &= h \cdot (y_1, g \cdot \phi_2) \\ &= (y_1, \phi_1). \end{aligned}$$

Thus, (y_1, ϕ_1) and (y_2, ϕ_2) are in the same G-orbit. Hence the induced map is injective. $\square$

Using this lemma, we get the following characterization of nilpotent orbits in G.

Theorem 3.1.13. *The equivalence classes of nilpotent orbits for strong involutions of* (G, γ) *are in bijection with* G*-orbits of pairs* $(y, X) \in G^\Gamma \times \mathfrak{g}^y m$ *where*

1. X *is an even nilpotent element of* $\mathfrak{g}^y$.

2. *For every* $\phi \colon SL(2, \mathbb{C}) \to G^y$ *such that* $d\phi(X_{SL(2),\theta}) = X$, *we have* $y^2\phi(-I) \in \mathcal{Z}(G)$.

The equivalence class containing (x, X) *corresponds to the* G*-orbit containing* $(x\phi(x_{SL(2)}, X)$, *where* $\phi \colon SL(2, \mathbb{C}) \to G^y$ *is any homomorphism satisfying* $d\phi(X_{SL(2),\theta}) = X$.

The equivalence classes of even nilpotent orbits for strong involutions of (G, γ) can be easily characterized in this bijection.

Corollary 3.1.14. *In the bijection in Theorem 3.1.13, the equivalence class of even nilpotent orbits for strong involutions of* (G, γ) *correspond to* G*-orbits of pairs* $(y, X) \in G^\Gamma \times \mathfrak{g}^y$, *where*

1. X *is an even nilpotent element of* $\mathfrak{g}^y$ *and*

2. $y^2 \in \mathcal{Z}(G)$.

Proof. By Proposition 2.1.4, if ϕ satisfies $d\phi(X_{SL(2),\theta}) = X$, then X is even if and only if $\phi(-I) = I$. □

Theorem 3.1.13 allows us to compute find all nilpotent orbits for strong involutions from the complex even nilpotent orbits and conjugacy classes of elements y so that $y^4 \in \mathcal{Z}(G)$.

3.2 Conjugacy classes in G^Γ

For this section, we will assume that G is a complex, simply connected, simple group. The automorphisms of G of finite order were classified in [9] and discussed in [6]. The classification is a generalization of the methods we discussed for elements of order 2. Let T be a torus with Lie algebra $\mathfrak{t}$ and let $(R, X, R^\vee, X^\vee)$ be a root system for G and let Π be a set of simple roots. Let X_i be a set of generators of the positive root spaces α_i. Let τ be an automorphism of the Dynkin diagram of order k. Then there is a unique automorphism γ_τ of G such that

$$\gamma_\tau(X_i) = X_{\tau i}$$
$$\gamma_\tau(H_i) = H_{\tau i}.$$

Every holomorphic automorphism of G is of the form $\mathrm{Ad}(g)\gamma_\tau$ for some element $g \in G$ and some diagram automorphism τ. For the remainder of this section, we fix a diagram automorphism τ of G of order one or two and let

$$\gamma = \gamma_\tau$$

be as above. Let G^Γ be the extended group for γ. For $\tilde{\beta} = (\beta, \epsilon) \in (\mathfrak{t}^\gamma)^* \times (\mathbb{Z}/2\mathbb{Z})$, let

$$\mathfrak{g}^{\tilde{\beta}} = \{X \in \mathfrak{g} |\, \mathrm{Ad}(t)x = \beta(t)X, \delta X = e^{\pi i \epsilon} X\}.$$

The Lie algebra $\mathfrak{g}$ decomposes into joint eigenspaces of T^γ and δ

$$\mathfrak{g} = \bigoplus_{\tilde{\beta}} \mathfrak{g}^{\tilde{\beta}}.$$

Let

$$\Pi_\tau = \{\beta_1, \beta_2, \ldots, \beta_r\}$$

denote a set of simple roots for the root system $\Delta(G^\gamma, T^\gamma)$. If τ is trivial, then let $\beta_0 = \alpha_0$ be the lowest root of $(\mathfrak{g}, \mathfrak{t})$. If τ is nontrivial, then let β_0 be the lowest short root for $(T^\gamma \subset G^\gamma)$ for which $\mathfrak{g}^{(\beta_0,1)} \neq 0$. Then

$$\beta_0 = -\sum_{i=1}^{r} m_i \beta_i$$

for some nonnegative integers m_i. We set $m_0 = 1$ so that $\sum_{i=1}^{r} m_i \beta_i = 0$.

Definition 3.2.1. The fundamental alcove of (G, τ) is the set of elements $H \in \mathfrak{t}^\gamma$ such that

1. $\beta_i(H) \geq 0$ for $1 \leq i \leq n$; and
2. $0 \leq k \sum_{i=1}^{r} m_i \beta_i(s) \leq 1$ (or equivalently $-1/k \leq \beta_0(H) \leq 0$).

The following theorem is proved in [1] and is closely related to results in [9].

Theorem 3.2.2. *Elements z of $G^\Gamma \backslash G$ for which $z^{lk} \in \mathcal{Z}(G)$ are each conjugate by G to a unique element* $\exp(2\pi i H)\delta$*, where H is in the fundamental alcove of* (G, τ).

This result allows one to determine the equivalence classes of holomorphic involutions for (G, γ), as well as the G-orbits of elements $y \in G^\Gamma \backslash G$ for which $y^4 \in \mathcal{Z}(G)$. Propositions 2.2.1 and 2.2.2 are special cases of this theorem when τ is the identity map.

The inner products between the elements β_0, β_1, ..., β_n are encoded in the affine Dynkin diagram for (G, γ), where, as in the usual Dynkin diagram, if

$$\langle \beta_i, \beta_j^\vee \rangle = -a,$$

there is an edge from β_i to β_j of multiplicity a. We also draw the numbers m_i at each vertex of the affine Dynkin diagram. When τ is the identity automorphism of the Dynkin diagram, the affine Dynkin diagram for (G, γ) is simply the extended Dynkin diagram of G. Affine Dynkin diagrams allow for a way to compute the fixed points of $y^{g\delta}$, when g is of the form of Theorem 3.2.2.

Theorem 3.2.3. *([6], Theorem 5.15) Let H be an element of the fundamental alcove for (G, τ) such that* $\exp(2\pi i H)$ *is of finite order. The Lie algebra* $\mathfrak{g}^{\exp(2\pi i H)\delta}$ *is a reductive Lie algebra with maximal torus* $\mathfrak{t}^{\gamma}$ *and simple roots*

$$\Pi_y = \{\beta_i \, | \beta_i(H) = 0 \; (i = 1, \ldots, r)\} \cup \{\beta_0 \; (\textit{if } \beta_0(H) = -1/k)\}.$$

Theorem 2.5.1 is a special case of this theorem.

Nilpotent orbits in $\mathfrak{g}^y$ are indexed by weighted Dynkin diagrams, which describe a semisimple element of $\mathfrak{g}^y$ as a linear combination of fundamental coweights of Π_y. The proposition below determines how to convert between an expression in the fundamental coweights for Π_y and the fundamental coweights of Π_τ.

Proposition 3.2.4. *Let H_y be an element of the fundamental alcove for (G, τ) such that* $\exp(2\pi i H_y)\delta$ *is of finite order. Let $y = \exp(2\pi i H_y)\delta$. Let*

$$\Pi_y = \{\omega_1, \ldots, \omega_s\}$$

be the simple roots for $(\mathfrak{g}^y, \mathfrak{t}^{\gamma})$ given by Theorem 3.2.3. Let B be the Cartan matrix for G^{γ} with entries

$$B_{kj} = \langle \omega_k, \omega_j^{\vee} \rangle$$

and let A be the matrix with entries

$$A_{kj} = \langle \beta_k, \omega_j^{\vee} \rangle.$$

Let H_ϕ be a semisimple element of a standard triple in $\mathfrak{g}^y$ that is contained in the maximal torus $\mathfrak{t}^{\gamma}$ and is dominant with respect to the simple roots Π_y. Let $\mathbf{b}$ be the vector with entries $b_j = \omega_j(H_\phi)$. Then the matrix $AB^{-1}\mathbf{b}$ has entries $a_j = \beta_j(H_\phi)$.

Proof. By Lemma 2.1.6, we have

$$H_\phi \in (\mathfrak{g}^y)'.$$

The torus $\mathfrak{t}^\gamma \cap (\mathfrak{g}^y)'$ has basis $\{\omega_1^\vee, \ldots, \omega_s^\vee\}$. Thus H_ϕ is of the form

$$H_\phi = \sum_{j=1}^{n} d_j \omega_j^\vee.$$

The coefficients b_j satisfy

$$\begin{aligned} b_k &= \langle \omega_k, \sum_{j=1}^{s} d_j \omega_j^\vee \rangle \\ &= \sum_{j=1}^{s} \langle \omega_i, \omega_j^\vee \rangle d_j \\ &= \sum_{j=1}^{s} B_{kj} d_j, \end{aligned}$$

and the vector $\mathbf{d}$ with entries d_j satisfies

$$\mathbf{b} = B\mathbf{d}.$$

The coefficients a_k satisfy

$$\begin{aligned} a_k &= \langle \beta_k, \sum_{j=1}^{s} d_j \omega_j^\vee \rangle \\ &= \sum_{j=1}^{s} \langle \beta_i, \omega_j^\vee \rangle d_j \\ &= \sum_{j=1}^{s} A_{kj} d_j. \end{aligned}$$

and so

$$\mathbf{a} = A\mathbf{d}.$$

Thus

$$\mathbf{a} = AB^{-1}\mathbf{b}.$$

$\square$

When $(G^y)'$ and G have the same rank, then the Dynkin diagram automorphism τ is trivial, and G^y has simple roots

$$\Pi_y = \{\beta_0, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_n\}.$$

In that case, the expression

$$\sum_{i=0}^{n} \beta_i(H) = 0$$

can be solved directly to express the fundamental coweights for Π_y in terms of those for Π_τ.

The following proposition describes the equivalence class of strong involutions of (G, τ) corresponding to a pair (y, X) as in Theorem 3.1.13.

Proposition 3.2.5. *Let y be an element of G^Γ such that $y^4 \in \mathcal{Z}(G)$. Then there exists a unique element H_y in the fundamental alcove of (G, τ) such that y is conjugate to* $\exp(2\pi i H_y)\delta$. *Let $\mathcal{O}$ be a nilpotent orbit of G^y. Let ϕ: $SL(2, \mathbb{C}) \to G$ be a homomorphism so that $d\phi(X_{SL(2),\theta}) = X$. Let H_ϕ be the dominant element of $\mathfrak{t}^\gamma$ conjugate to $d\phi(H_{SL(2)}$ (so that the values $\beta_i(H_\phi)$ for $\beta_i \in \Pi_y$ are the values in the weighted Dynkin diagram for G). Then $y\phi(x_{SL(2)}^{-1})$ is conjugate by G to* $\exp(\pi i H_x)\delta$ *where*

$$H_x = 2H_y + H_\phi/2.$$

Proof. The existence and uniqueness of H_y follows from Theorem 3.2.2. Let g be an element of G such that

$$g \cdot y = \exp(2\pi i H_y).$$

The element

$$x_{SL(2)}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is conjugate in $SL(2,\mathbb{C})$ to

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \exp(\pi i H_{SL(2)}/2).$$

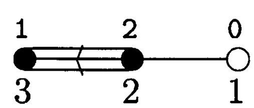
As the image of $g \cdot \phi$ is in $G^{g \cdot y}$, it follows that $g \cdot \phi(x_{SL(2)}^{-1})$ is conjugate by $G^{g \cdot y}$ to $g \cdot \phi(\pi i H_{SL(2)}/2)$. Therefore, $y\phi(x_{SL(2)}^{-1})$ is conjugate by G to

$$\exp(2\pi i H_y) \exp(\pi i H_{SL(2)}/2) = \exp(\pi i(2H_y + H_\phi/2)).$$

$\square$

3.3 Equivalence Classes of Nilpotent Orbits for Strong Involutions of G_2

Let G be the simply connected, connected group with Lie algebra G_2. We now discuss the equivalence classes of nilpotent orbits for strong involutions of G in the context of Theorem 3.1.13. Since G_2 has no diagram automorphisms, there is only one inner class of automorphisms of G. Let τ be the identity automorphism of the Dynkin diagram of G, so that $\gamma = \gamma_\tau$ is the identity automorphism of G. The labeled extended Dynkin diagram of G is



.

From this diagram and Proposition 2.2.1, the center of G is trivial.

From Theorem 3.2.2, every conjugacy class of elements $y \in G$ satisfying $y^4 = 1$ contains a unique element of the form $\exp(2\pi i H_y)$, where H_y is one of 0, $H_1/4$, $H_2/4$, and $H_2/2$. Theorem 3.2.3 gives the centralizers of these as in the following table.

H_y	G^y	Simple Roots of G^y
0	G_2	α_1, α_2
$H_1/4$	A_1	α_2
$H_2/4$	A_1	α_1
$H_2/2$	$A_1 \times A_1$	α_0, α_1

For each even nilpotent orbit $\mathcal{O}'$ in $(\mathfrak{g}^y)$, let H_ϕ be the element of $\mathfrak{t}^\gamma$ dominant with respect to the simple roots Π_y from Theorem 3.2.3 and an element of a standard triple for $\mathcal{O}'$. The Dynkin diagram for $\mathcal{O}$ contains the values $\alpha_i(H_\phi)$ whenever $\alpha_i \in \Pi_y$.

When $H_y = 0$ and $H_y = H_2/2$ the groups G^y are semisimple, and two of the three values $\alpha_j(H_\phi)$ are in the Dynkin diagram for $\mathcal{O}$ and we can fill in the remaining root $\alpha_i(H_\phi)$ for the remaining root α_i by solving

$$\alpha_0(H_\phi) + 3\alpha_1(H_\phi) + 2\alpha_1(H_\phi) = 0$$

for $\alpha_i(H)$.

Next we consider the case when $H_y = H_1/4$. In that case, G^y has Dynkin diagram A_1, so the Cartan matrix for the simple roots Π_y is simply

$$B = (2).$$

The matrix A with entry $A_{i1} = \langle \alpha_i, \alpha_2 \rangle$, which is a row of the extended Cartan matrix for G_2 is given by

$$A = \begin{pmatrix} -1 & 2 \end{pmatrix}.$$

Thus

$$AB^{-1} = \begin{pmatrix} -1/2 & 1 \end{pmatrix}.$$

The even nilpotent orbits in type A_1 have Dynkin diagrams with either a single 2 or a single 0. For the nonzero orbit, the element H_ϕ satisfies

$$\begin{aligned} \begin{pmatrix} \alpha_1(H_\phi) & \alpha_2(H_\phi) \end{pmatrix} &= \begin{pmatrix} -1/2 & 1 \end{pmatrix} (2) \\ &= \begin{pmatrix} -1 & 2 \end{pmatrix}. \end{aligned}$$

The case when $H_y = H_2/4$ follows similarly.

Below we have included written the values $\alpha_i(H_\phi)$ for each of the elements H_ϕ for even orbits in Lie algebras $\mathfrak{g}^y$.

H_y	G^y	$\alpha_1(H_\phi)$	$\alpha_2(H_\phi)$	$\alpha_0(H_\phi)$
0	G_2	0	0	0
		0	2	-4
		2	2	-10
$H_1/4$	A_1	0	0	0
		-1	2	-1
$H_2/4$	A_1	0	0	0
		2	-3	0
$H_2/2$	$A_1 \times A_1$	0	0	0
		2	-3	0
		0	-1	2
		2	-4	2

In the correspondence in Theorem 3.1.13, the nilpotent orbit $\mathcal{O}$ corresponds to an equivalence class of nilpotent orbits for the equivalence class of strong involutions $G \cdot y\phi(x_{SL(2)}^{-1})$ if $y^2\phi(-I)$ is central. By proposition 3.2.5, the element $y\phi(x_{SL(2)}^{-1})$ is conjugate to $\exp(\pi i H_x)$, where $H_x = 2H_y + H_\phi/2$. Then $y^2\phi(-I)$ is central if and only H_x is an integral combination of fundamental coweights for Π. We include the values $\alpha_i(H_x)$ for $0 \le i \le 2$ in the Table 3.1. When H_x is an integral combination of fundamental coweights for Π, we also include the element H_x' for which $\exp(\pi i H_x')$ is conjugate to $\exp(\pi i H_x)$ and H_x' in the fundamental alcove for G, which is found by the algorithm in Proposition 2.3.4.

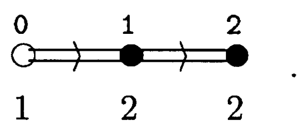
Table 3.1:

Data on Nilpotent Orbits for Strong Involutions of G_2								
H_y	G^y	$\alpha_1(H_\phi)$	$\alpha_2(H_\phi)$	$\alpha_0(H_\phi)$	$\alpha_1(H_x)$	$\alpha_2(H_x)$	$\alpha_0(H_x)$	H_x'
0	G_2	0	0	0	0	0	0	0
		0	2	-4	0	1	-2	H_2
		2	2	-10	1	1	-5	H_2
$H_1/4$	A_1	0	0	0	1/2	0	-3/2	
		-1	2	-1	0	1	-2	H_2
$H_2/4$	A_1	0	0	0	0	1/2	-1	
		2	-3	0	1	-1	-1	H_2
$H_2/2$	$A_1 \times A_1$	0	0	0	0	1	-2	H_2
		2	-3	0	1	-1/2	-2	
		0	-1	2	0	1	-2	
		2	-4	2	1	-1	-1	H_2

This chart identifies the 7 equivalence classes nilpotent elements for strong involutions of G_2, including a single one associated to the trivial strong involution of G, and 6 associated to the other the equivalence class of strong involutions.

3.4 Equivalence classes of nilpotent orbits for A_4 with the nontrivial involution

Let $G = SL(5, \mathbb{C})$. Let τ be the non-trivial involution of G and let $\gamma = \gamma_\tau$ be the associated automorphism of G. We will discuss equivalence classes of nilpotent orbits for strong involutions of (G, γ) in the context of Theorem 3.1.13. The affine Dynkin diagram for (G, τ) is



There is only one equivalence class of strong involutions for G since the only element $H \in \mathfrak{t}^\gamma$ satisfying the conditions in Theorem 3.2.2 is 0. This corresponds by Theorem 3.1.2 to the real group $SL(5, \mathbb{R})$.

Let M_1 and M_2 be the fundamental coweights for the simple Π_τ of $(\mathfrak{g}^\gamma, \mathfrak{t}^\gamma)$. The coweights are given by

$$\begin{aligned} M_1 &= H_1 + H_4 \\ M_2 &= H_2 + H_3. \end{aligned}$$

The conjugacy classes of elements $y \in G^\Gamma$ for which of $y^4 \in \mathcal{Z}(G)$ and $y \in G^\Gamma \backslash G$ each contain exactly one element of the form $\exp(2\pi i H_y)\delta$, where $H_y \in \mathfrak{g}^\gamma$, the element $4H_y$ is a linear combination of M_1 and M_2, and

$$\beta_1(H_y) + \beta_2(H_y) \leq 1/2.$$

The possible values of H_y are 0, $M_1/4$, and $M_2/4$. The centralizers of these are listed in the table below.

H_y	G^y	Simple Roots of G^y
0	B_2	β_1, β_2
$H_1/4$	$A_1 \times A_1$	β_0, β_2
$H_2/4$	B_2	β_0, β_1

For each even nilpotent orbit $\mathcal{O}'$ in $(\mathfrak{g}^y)$, let H_ϕ be the element of $\mathfrak{t}^\gamma$ dominant with respect to the simple roots Π_y from Theorem 3.2.3 and an element of a standard triple for $\mathcal{O}'$. The Dynkin diagram for $\mathcal{O}$ contains the values $\beta_i(H_\phi)$ whenever $\beta_i \in \Pi_y$. The remaining value $\beta_i(H_\phi)$ can be calculated from the relation

$$\beta_0(H_\phi) + 2\beta_1(H_\phi) + 2\beta_2(H_\phi) = 0.$$

Below we have included the values $\beta_i(H_\phi)$ for each of the even nilpotent orbit of each Lie algebra $\mathfrak{g}^y$ and each root β_i with $0 \leq i \leq 2$.

H_y	G^y	$\beta_0(H_\phi)$	$\beta_1(H_\phi)$	$\beta_2(H_\phi)$
0	B_2	0	0	0
		-4	2	0
		-8	2	2
$H_1/4$	$A_1 \times A_1$	0	0	0
		2	-4	0
		0	-2	2
		2	-3	2
$H_2/4$	B_2	0	0	0
		2	0	-1
		2	2	-3

In the correspondence in Theorem 3.1.13, the nilpotent orbit $\mathcal{O}$ corresponds to an equivalence class of nilpotent orbits for the equivalence class of strong involutions $G \cdot y\phi(x_{SL(2)}^{-1})$, whenever $y^2\phi(-I)$ is central. By Proposition 3.2.5, $y\phi(x_{SL(2)}^{-1})$ is conjugate by G to $\exp(\pi i H_x)$, where

$$H_x = 2H_y + H_\phi/2.$$

From this, it follows that

$$y^2\phi(-I) = \exp(2\pi i H_x)$$

and $y^2\phi(-I)$ is central if and only if H_x is an integral combination of fundamental coweights for Π.

Table 3.2 includes the coefficients of the element H_x in terms of the fundamental coweights for Π_τ.

The conversion from fundamental coweights for Π_τ to fundamental coweights for Π is found in [6], chapter 10, Example 2 after Lemma 5.11. In this case,

$$\begin{aligned} M_1 &= H_1 + H_4 \\ M_2 &= H_2 + H_3. \end{aligned}$$

Hence, an element of $\mathfrak{t}^\gamma$ is an integral combination of fundamental coweights for Π if and only if it is an integral combination of coweights for Π_τ.

By Theorem 3.1.13, Proposition 3.1.8 and Theorem 3.1.5, The 7 pairs (H_y, H_ϕ) for which H_x is an integral combination of fundamental coweights for Π_τ correspond to the seven nilpotent orbits in $SL(5, \mathbb{R})$.

Table 3.2:

Data on Nilpotent Orbits for Strong Involutions of (SL_5, γ)							
H_y	$G^{\exp(2\pi i H_y)}$	$\beta_0(H_\phi)$	$\beta_1(H_\phi)$	$\beta_2(H_\phi)$	$\beta_0(H_x)$	$\beta_1(H_x)$	$\beta_2(H_x)$
0	B_2	0	0	0	0	0	0
		-4	2	0	-2	1	0
		-8	2	2	-4	1	1
$H_1/4$	$A_1 \times A_1$	0	0	0	-1	1/2	0
		2	-1	0	0	0	0
		0	-2	2	-1	-1/2	1
		2	-3	2	0	-1	1
$H_2/4$	B_2	0	0	0	-1	0	1/2
		2	0	-1	0	0	0
		2	2	-3	0	1	-1

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